# 2020 CFA®

PROGRAM EXAM PREP

SchweserNotes™ Level I

## Ethical and Professional Standards and Quantitative Methods

eBook 1

KAPLAN® SCHWESER

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# Kaplan Schweser's Path to Success

## Level I CFA® Exam

### Welcome

As the head of Advanced Designations at Kaplan Schweser, I am pleased to have the opportunity to help you prepare for the CFA® exam. Kaplan Schweser has decades of experience in delivering the most effective CFA exam prep products in the market and I know you will find them to be invaluable in your studies.

Our products are designed to be an integrated study solution across print and digital media to provide you the best learning experience, whether you are studying with a physical book, online, or on your mobile device.

Our core product, the SchweserNotes™, addresses all of the Topics, Study Sessions, Readings, and LOS in the CFA curriculum. Each reading in the SchweserNotes has been broken into smaller, bite-sized modules with Module Quizzes interspersed throughout to help you continually assess your comprehension. Topic Assessments appear at the end of each Topic to help you assess your knowledge of the material before you move on to the next section.

All purchasers of the SchweserNotes receive online access to the Kaplan Schweser online platform (our learning management system or LMS) at www.Schweser.com. In the LMS, you will see a dashboard that tracks your overall progress and performance and also includes an Activity Feed, which provides structure and organization to the tasks required to prepare for the CFA exam. You also have access to the SchweserNotes, Module Quizzes, and Topic Assessments content as well as the Video Lectures (if purchased), which contain a short video that complements each module in the SchweserNotes. Look for the icons indicating where video content, Module Quizzes, and Topic Assessments are available online. I strongly encourage you to enter your Module Quiz and Topic Assessment answers online and use the dashboard to track your progress and stay motivated.

Again, thank you for trusting Kaplan Schweser with your CFA exam preparation. We're here to help you throughout your journey to become a CFA charterholder.

Regards,
Derek Burkett, CFA, FRM, CAIA
Vice President (Advanced Designations)

**Contact us for questions about your study package, upgrading your package, purchasing additional study materials, or for additional information:**

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# WELCOME TO THE 2020 SCHWESERNOTES™

Thank you for trusting Kaplan Schweser to help you pass the Level I CFA exam. You have made an exceptionally good decision, and we congratulate you for taking on the challenge of earning your CFA charter.

Your first step should be to view the "How to Pass the Level I CFA Exam" video (available in your Resource Library), in which I explain the structure of the Level I exam, the format of exam questions, and topic area exam weights. We also provide advice on interpreting the (500+) Level I CFA Learning Outcome Statements (LOS), how to create an effective study plan, and study techniques based on research in learning science. Understanding the exact nature of the challenge you have taken on is an important first step in preparing to pass the Level I CFA exam.

The next step is to study and learn the material required for the exam. The best time to begin that study is today (regardless of when you are reading this). Less than 40% of those who register for a Level I exam pass the exam (including re-takers). For many, passing the exam is a formidable challenge. One of reasons candidates give most frequently for failing is "not starting early enough."

Begin your study today with Study Session 2, Quantitative Methods, and progress through Study Session 19, Portfolio Management, over the coming months. Complete your initial study of the Level I CFA curriculum with Study Session 1, Ethical and Professional Standards. While the SchweserNotes provide an excellent summary of the required material for Study Session 1, we strongly recommend that, for this material, all candidates also study the CFA Institute *Code of Ethics and Standards of Professional Conduct* (Readings 2 and 3 in the Level I CFA Curriculum, Volume 1) at this point and again shortly before the exam.

It is very important to finish your initial study of the entire curriculum at least one month (earlier if possible) prior to your exam date to allow sufficient time for practice and targeted review. During this period, you should take all the Schweser Practice Exams. Two weeks prior to the exam you should take the Schweser Mock Exam, which is offered by CFA Societies in well over 100 locations worldwide, as well as online for those who can't make it to a live Schweser Mock. This final review period is when you will get a clear indication of how effective your study has been and which topic areas require significant additional review on your part. Practice answering exam-like questions across all topic areas and working on your exam timing will be important determinants of your success on exam day.

Finally, I would like to thank my assistant, Craig Prochaska, CFA, who has been invaluable in the preparation of all our Level I study materials and candidate support for 14 years. Craig and I will be answering your questions and supporting your study throughout the exam season.

Best regards,

Doug Van Eaton, PhD, CFA
SVP for CFA Education
Kaplan Schweser

## LEARNING OUTCOME STATEMENTS (LOS)

# STUDY SESSION 1

*This topical coverage corresponds with the following CFA Institute assigned reading:*

## 1. Ethics and Trust in the Investment Profession

The candidate should be able to:

a. explain ethics. (page 1)
b. describe the role of a code of ethics in defining a profession. (page 2)
c. describe professions and how they establish trust. (page 2)
d. describe the need for high ethical standards in investment management. (page 3)
e. explain professionalism in investment management. (page 3)
f. identify challenges to ethical behavior. (page 4)
g. distinguish between ethical and legal standards. (page 4)
h. describe and apply a framework for ethical decision making. (page 5)

*This topical coverage corresponds with the following CFA Institute assigned reading:*

## 2. Code of Ethics and Standards of Professional Conduct

The candidate should be able to:

a. describe the structure of the CFA Institute Professional Conduct Program and the process for the enforcement of the Code and Standards. (page 9)
b. state the six components of the Code of Ethics and the seven Standards of Professional Conduct. (page 10)
c. explain the ethical responsibilities required by the Code and Standards, including the sub-sections of each Standard. (page 11)

*This topical coverage corresponds with the following CFA Institute assigned reading:*

## 3. Guidance for Standards I–VII

The candidate should be able to:

a. demonstrate the application of the Code of Ethics and Standards of Professional Conduct to situations involving issues of professional integrity. (page 17)
b. distinguish between conduct that conforms to the Code and Standards and conduct that violates the Code and Standards. (page 17)
c. recommend practices and procedures designed to prevent violations of the Code of Ethics and Standards of Professional Conduct. (page 17)

*This topical coverage corresponds with the following CFA Institute assigned reading:*

## 4. Introduction to the Global Investment Performance Standards (GIPS®)

The candidate should be able to:

a. explain why the GIPS standards were created, what parties the GIPS standards apply to, and who is served by the standards. (page 47)
b. explain the construction and purpose of composites in performance reporting. (page 48)
c. explain the requirements for verification. (page 48)

*This topical coverage corresponds with the following CFA Institute assigned reading:*

## 5. Global Investment Performance Standards (GIPS®)

The candidate should be able to:

a. describe the key features of the GIPS standards and the fundamentals of compliance. (page 53)
b. describe the scope of the GIPS standards with respect to an investment firm's definition and historical performance record. (page 55)

c. explain how the GIPS standards are implemented in countries with existing standards for performance reporting and describe the appropriate response when the GIPS standards and local regulations conflict. (page 56)
d. describe the nine major sections of the GIPS standards. (page 56)

# STUDY SESSION 2

*This topical coverage corresponds with the following CFA Institute assigned reading:*

## 6. The Time Value of Money

The candidate should be able to:

a. interpret interest rates as required rates of return, discount rates, or opportunity costs. (page 73)

b. explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk. (page 74)

c. calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding. (page 74)

d. solve time value of money problems for different frequencies of compounding. (page 76)

e. calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows. (page 77)

f. demonstrate the use of a time line in modeling and solving time value of money problems. (page 89)

*This topical coverage corresponds with the following CFA Institute assigned reading:*

## 7. Statistical Concepts and Market Returns

The candidate should be able to:

a. distinguish between descriptive statistics and inferential statistics, between a population and a sample, and among the types of measurement scales. (page 101)

b. define a parameter, a sample statistic, and a frequency distribution. (page 103)

c. calculate and interpret relative frequencies and cumulative relative frequencies, given a frequency distribution. (page 105)

d. describe the properties of a data set presented as a histogram or a frequency polygon. (page 106)

e. calculate and interpret measures of central tendency, including the population mean, sample mean, arithmetic mean, weighted average or mean, geometric mean, harmonic mean, median, and mode. (page 109)

f. calculate and interpret quartiles, quintiles, deciles, and percentiles. (page 113)

g. calculate and interpret 1) a range and a mean absolute deviation and 2) the variance and standard deviation of a population and of a sample. (page 114)

h. calculate and interpret the proportion of observations falling within a specified number of standard deviations of the mean using Chebyshev's inequality. (page 119)

i. calculate and interpret the coefficient of variation. (page 120)

j. explain skewness and the meaning of a positively or negatively skewed return distribution. (page 121)

k. describe the relative locations of the mean, median, and mode for a unimodal, nonsymmetrical distribution. (page 121)

l. explain measures of sample skewness and kurtosis. (page 123)

m. compare the use of arithmetic and geometric means when analyzing investment returns. (page 125)

*This topical coverage corresponds with the following CFA Institute assigned reading:*

## 8. Probability Concepts

The candidate should be able to:

a. define a random variable, an outcome, an event, mutually exclusive events, and exhaustive events. (page 133)
b. state the two defining properties of probability and distinguish among empirical, subjective, and a priori probabilities. (page 134)
c. state the probability of an event in terms of odds for and against the event. (page 134)
d. distinguish between unconditional and conditional probabilities. (page 135)
e. explain the multiplication, addition, and total probability rules. (page 136)
f. calculate and interpret 1) the joint probability of two events, 2) the probability that at least one of two events will occur, given the probability of each and the joint probability of the two events, and 3) a joint probability of any number of independent events. (page 136)
g. distinguish between dependent and independent events. (page 139)
h. calculate and interpret an unconditional probability using the total probability rule. (page 140)
i. explain the use of conditional expectation in investment applications. (page 141)
j. explain the use of a tree diagram to represent an investment problem. (page 142)
k. calculate and interpret covariance and correlation and interpret a scatterplot. (page 143)
l. calculate and interpret the expected value, variance, and standard deviation of a random variable and of returns on a portfolio. (page 148)
m. calculate and interpret covariance given a joint probability function. (page 149)
n. calculate and interpret an updated probability using Bayes' formula. (page 150)
o. identify the most appropriate method to solve a particular counting problem and solve counting problems using factorial, combination, and permutation concepts. (page 151)

## STUDY SESSION 3

*This topical coverage corresponds with the following CFA Institute assigned reading:*

### 9. Common Probability Distributions

The candidate should be able to:

a. define a probability distribution and distinguish between discrete and continuous random variables and their probability functions. (page 163)
b. describe the set of possible outcomes of a specified discrete random variable. (page 163)
c. interpret a cumulative distribution function. (page 165)
d. calculate and interpret probabilities for a random variable, given its cumulative distribution function. (page 165)
e. define a discrete uniform random variable, a Bernoulli random variable, and a binomial random variable. (page 167)
f. calculate and interpret probabilities given the discrete uniform and the binomial distribution functions. (page 167)
g. construct a binomial tree to describe stock price movement. (page 170)
h. define the continuous uniform distribution and calculate and interpret probabilities, given a continuous uniform distribution. (page 171)
i. explain the key properties of the normal distribution. (page 174)
j. distinguish between a univariate and a multivariate distribution and explain the role of correlation in the multivariate normal distribution. (page 175)
k. determine the probability that a normally distributed random variable lies inside a given interval. (page 176)
l. define the standard normal distribution, explain how to standardize a random variable, and calculate and interpret probabilities using the standard normal distribution. (page 178)
m. define shortfall risk, calculate the safety-first ratio, and select an optimal portfolio using Roy's safety-first criterion. (page 182)
n. explain the relationship between normal and lognormal distributions and why the lognormal distribution is used to model asset prices. (page 184)
o. distinguish between discretely and continuously compounded rates of return and calculate and interpret a continuously compounded rate of return, given a specific holding period return. (page 185)
p. explain Monte Carlo simulation and describe its applications and limitations. (page 186)
q. compare Monte Carlo simulation and historical simulation. (page 187)

*This topical coverage corresponds with the following CFA Institute assigned reading:*

### 10. Sampling and Estimation

The candidate should be able to:

a. define simple random sampling and a sampling distribution. (page 196)
b. explain sampling error. (page 196)
c. distinguish between simple random and stratified random sampling. (page 197)
d. distinguish between time-series and cross-sectional data. (page 197)
e. explain the central limit theorem and its importance. (page 198)
f. calculate and interpret the standard error of the sample mean. (page 198)
g. identify and describe desirable properties of an estimator. (page 200)
h. distinguish between a point estimate and a confidence interval estimate of a population parameter. (page 202)

i. describe properties of Student's t-distribution and calculate and interpret its degrees of freedom. (page 203)
j. calculate and interpret a confidence interval for a population mean, given a normal distribution with 1) a known population variance, 2) an unknown population variance, or 3) an unknown population variance and a large sample size. (page 205)
k. describe the issues regarding selection of the appropriate sample size, data-mining bias, sample selection bias, survivorship bias, look-ahead bias, and time-period bias. (page 209)

*This topical coverage corresponds with the following CFA Institute assigned reading:*

### 11. Hypothesis Testing

The candidate should be able to:

a. define a hypothesis, describe the steps of hypothesis testing, and describe and interpret the choice of the null and alternative hypotheses. (page 218)
b. distinguish between one-tailed and two-tailed tests of hypotheses. (page 219)
c. explain a test statistic, Type I and Type II errors, a significance level, and how significance levels are used in hypothesis testing. (page 221)
d. explain a decision rule, the power of a test, and the relation between confidence intervals and hypothesis tests. (page 223)
e. distinguish between a statistical result and an economically meaningful result. (page 226)
f. explain and interpret the p-value as it relates to hypothesis testing. (page 227)
g. identify the appropriate test statistic and interpret the results for a hypothesis test concerning the population mean of both large and small samples when the population is normally or approximately normally distributed and the variance is 1) known or 2) unknown. (page 228)
h. identify the appropriate test statistic and interpret the results for a hypothesis test concerning the equality of the population means of two at least approximately normally distributed populations, based on independent random samples with 1) equal or 2) unequal assumed variances. (page 233)
i. identify the appropriate test statistic and interpret the results for a hypothesis test concerning the mean difference of two normally distributed populations. (page 236)
j. identify the appropriate test statistic and interpret the results for a hypothesis test concerning 1) the variance of a normally distributed population, and 2) the equality of the variances of two normally distributed populations based on two independent random samples. (page 239)
k. formulate a test of the hypothesis that the population correlation coefficient equals zero and determine whether the hypothesis is rejected at a given level of significance. (page 244)
l. distinguish between parametric and nonparametric tests and describe situations in which the use of nonparametric tests may be appropriate. (page 245)

The following is a review of the Ethical and Professional Standards principles designed to address the learning outcome statements set forth by CFA Institute. Cross-Reference to CFA Institute Assigned Reading #1.

# READING 1: ETHICS AND TRUST IN THE INVESTMENT PROFESSION

**Study Session 1**

## EXAM FOCUS

From this reading, candidates should learn the definitions of ethics and ethical behavior presented by the authors and the arguments presented for having a code of ethics and following ethical principles. Additionally, the arguments for integrating ethics into the decision-making process include testable material.

## MODULE 1.1: ETHICS AND TRUST

Video covering this content is available online.

### LOS 1.a: Explain ethics.

*CFA® Program Curriculum, Volume 1, page 7*

**Ethics** can be described as a set of shared beliefs about what is good or acceptable behavior and what is bad or unacceptable behavior. Ethical conduct has been described as behavior that follows moral principles and is consistent with society's ethical expectations.

Ethical conduct has also been described as conduct that improves outcomes for **stakeholders**, who are people directly or indirectly affected by the conduct. Examples of stakeholders in the case of investment professionals include their clients, coworkers, employers, and the investment profession as a whole. Some decisions may bring positive results for you, but negative consequences for a stakeholder, such as a coworker. Ethical conduct is behavior that balances your self-interest with the impact on others.

### LOS 1.b: Describe the role of a code of ethics in defining a profession.

*CFA® Program Curriculum, Volume 1, page 9*

A **code of ethics** is a written set of moral principles that can guide behavior by describing what is considered acceptable behavior. Having a code of ethics is a way to communicate the values, principles, and expectations of an organization or other group of people and provides a general guide to what constitutes acceptable behavior. Some codes of ethics include a set of rules or standards that require some minimum level of ethical behavior.

A **profession** refers to a group of people with specialized skills and knowledge who serve others and agree to behave in accordance with a code of ethics. A professional code of ethics is a way for a profession to communicate to the public that its members will use their knowledge and skills to serve their clients in an honest and ethical manner.

### LOS 1.c: Describe professions and how they establish trust.

*CFA® Program Curriculum, Volume 1, page 9*

A **profession** is an occupational group (e.g., doctors or lawyers) that has requirements of specialized expert knowledge, and often a focus on ethical behavior and service to the larger community or society. Additionally, a profession may have the following characteristics:

- A code and standards for professional behavior.
- A regulatory body to enforce rules concerning professional behavior and monitor the ethical behavior of members.
- A focus on the needs of their clients (e.g., students, patients).
- A focus on service to society.
- A requirement to put client interests first.
- A focus on or requirement for continuing education.

Ways that professions establish trust include:

- Requiring high standards of expertise, knowledge, and skill.
- Establishing standards of ethical behavior.
- Monitoring professional conduct.
- Encouraging continuing education to maintain and increase competence.
- Being focused on clients' needs.
- Mentoring and inspiring others in the profession.

### LOS 1.d: Describe the need for high ethical standards in investment management.

*CFA® Program Curriculum, Volume 1, page 12*

Investment professionals have a special responsibility because they are entrusted with their clients' wealth. The responsibility to use their specialized knowledge and skills to both protect and grow client assets makes high ethical standards all the more important. Investment advice and management are intangible products, making quality and value received more difficult to evaluate than for tangible products such as a laptop computer or a restaurant meal. For this reason, trust in investment professionals takes on an even greater importance than in many other businesses.

Failure to act in a highly ethical manner can damage not only client wealth but also impede the success of investment firms and investment professionals because potential investors will be less likely to use their services.

Unethical behavior by financial services professionals can have negative effects for society as a whole. The financial services industry serves as an intermediary between savers and those seeking financing for their business activities. A lack of trust in financial advisors will reduce the funds entrusted to them and increase the cost of raising capital for business investment and growth. When investors cannot rely on the information they receive from financial services professionals, this adds another layer of risk on top of the investment risks that investors face. Even the perception of additional risk will reduce the amounts invested and increase the returns required to attract investor capital.

In addition to reducing the amount of investment overall, unethical behavior—such as providing incomplete, misleading, or false information to investors—can affect the allocation of the capital that is raised. Misallocation of capital to businesses other than those with the most potential for growth and societal benefit reduces the growth of an economy and the

well-being of its people. When the allocation of investment capital is constrained or inefficient, the negative consequences extend to all the participants in an economy.

### LOS 1.e: Explain professionalism in investment management.

*CFA® Program Curriculum, Volume 1, page 12*

Because clients of investment professionals rely on their expertise, judgment, and ethical principles, many of the characteristics of a profession we have described apply.

Ethical principles are of great importance because clients often do not have significant knowledge about financial securities, fee structures, or sources of potential bias in investment recommendations. Currently, some financial professionals are held to a **suitability standard**, while others are held to a **fiduciary standard**. *Suitability* refers to the match between client return requirements and risk tolerances and the characteristics of the securities recommended. A fiduciary standard is stronger, requiring professionals to use their knowledge and expertise to act in the best interests of the client.

### LOS 1.f: Identify challenges to ethical behavior.

*CFA® Program Curriculum, Volume 1, page 15*

One challenge to ethical behavior is that individuals tend to overrate the ethical quality of their behavior on a relative basis and overemphasize the importance of their own personal traits in determining the ethical quality of their behavior.

It is claimed that external or **situational influences** are a more important determinant of the ethical quality of behavior than internal (personal) traits that influence behavior. One situational influence is social pressure from others. Loyalty to an employer, supervisor, organization, or coworkers can cause individuals to act in unethical ways as they place more importance on their self-interest and short-term results than on longer-term results and the ethical quality of their decisions and behavior. The prospect of acquiring more money or greater prestige can cause individuals to engage in unethical behavior.

Firms with strict rules-based compliance procedures run the risk of fostering a culture that is so focused on adhering to compliance rules that individuals only ask themselves what they *can* do. The question of what behavior they *should* engage in, based on ethical principles and longer-term results, is often not addressed in such situations.

### LOS 1.g: Distinguish between ethical and legal standards.

*CFA® Program Curriculum, Volume 1, page 17*

Not all unethical actions are illegal, and not all illegal actions are unethical. In some places it may be illegal to report one's employer's actions against the best interests of clients by sharing what is considered private company information with authorities, but doing so may be considered ethical "whistle-blowing" behavior by some. Acts of civil disobedience that are illegal are also considered by many to be ethical behavior. On the other hand, recommending investment in a relative's firm without disclosure may not be illegal, but would be considered unethical by many.

Ethical principles often set a higher standard of behavior than laws and regulations. New laws and regulations often result from recent instances of what is perceived to be unethical

behavior. Just as the Securities Act of 1933, the Glass-Steagall Act, and the Securities Exchange Act of 1934 followed the perceived bad behavior by investment professionals and bankers leading to the 1929 market crash, the Sarbanes-Oxley laws followed the accounting scandals at Enron and Worldcom, and the Dodd-Frank Act followed the 2008 financial crisis. New laws and regulations can create opportunities for different unethical behavior. In general, ethical decisions require more judgment and consideration of the impact of behavior on many stakeholders compared to legal decisions.

### LOS 1.h: Describe and apply a framework for ethical decision making.

*CFA® Program Curriculum, Volume 1, page 19*

Ethical decisions will be improved when ethics are integrated into a firm's decision making process. This will allow decision makers and teams to consider alternative actions as well as shorter- and longer-term consequences from various perspectives, improving the ethical aspects of their decisions. To do this it is first necessary that the firm adopt a code of ethics to guide the process.

Such integration provides an opportunity to teach, practice, and reinforce ethical decision making. This is an important part of developing an ethical culture. The support of senior management for integrating ethics into the decision-making process is also very important in developing a culture and processes that will result in ethical decision making.

Using a framework for ethical decision making helps individuals identify the important issues involved, examine these issues from multiple perspectives, develop the necessary judgment and decision making skills required, and avoid unanticipated ethical consequences.

The following ethical decision-making framework is presented in the Level I CFA curriculum:[1]

- Identify: Relevant facts, stakeholders and duties owed, ethical principles, conflicts of interest.
- Consider: Situational influences, additional guidance, alternative actions.
- Decide and act.
- Reflect: Was the outcome as anticipated? Why or why not?

In the first step, decision makers need to identify the facts they have to work with, and the facts they would like to have, before making a decision. Stakeholders—those affected by the decision—must be identified. These stakeholders may include the employer, clients, coworkers, self, family, and others in the industry, and the duties to each stakeholder should be identified. This part of the process will also help in explicitly identifying potential conflicts of interest among the various stakeholders. At this point the decision makers should be able to identify the ethical principles involved in the decision, although greater clarity about those may also be gained throughout the process.

In the second step, the framework suggests situational factors that may influence decision makers should be identified and considered along with any personal biases that may come into play. At this point, decision makers may seek outside guidance which can come from a mentor, colleagues, or friends who have shown good judgment in the past. Guidance may also be sought from the firm's legal and compliance departments. This guidance from alternative sources will help to provide a variety of perspectives from which the decision

under consideration can be viewed, as well as help in developing alternatives that should be considered.

Finally, the alternative actions that have been identified are all considered, taking into account both the short-term and long-term effects of each alternative action and any potential but unanticipated ethical implications.

In the final step, decision makers should evaluate the outcomes of the actions that were taken. In particular, they should consider whether the decisions had their intended results and whether appropriate consideration was given to ethical principles, situational influences, and duties to clients and other stakeholders.

## MODULE QUIZ 1.1

*To best evaluate your performance, enter your quiz answers online.*

1. A code of ethics:
   A. is a personal view of acceptable behavior.
   B. encompasses current “best practices.”
   C. specifies a minimum level of acceptable conduct.
2. A professional code of conduct:
   A. can increase public trust in the profession.
   B. guarantees that members will adhere to a minimum level of ethical conduct.
   C. includes standards that provide guidance for specific behaviors.
3. Situational factors that influence ethical behavior are *least likely* to include:
   A. social pressure.
   B. large financial rewards.
   C. a lack of ethical principles.
4. Compared to complying with laws and regulations, complying with a code of ethics:
   A. is considered a lower standard.
   B. often involves more judgment.
   C. includes compliance with all laws and regulations.
5. Employing a framework for decision making that includes the ethical aspects of the decision is *most likely* to:
   A. lead to higher profits.
   B. avoid any unintended ethical consequences of decisions.
   C. balance the interests of various stakeholders.

## KEY CONCEPTS

### LOS 1.a

Ethical behavior is that which conforms to a set of rules and moral principles based on shared beliefs about what behavior is acceptable and what behavior is unacceptable.

### LOS 1.b

A professional code of ethics is a way for a profession to communicate to the public that its members will use their knowledge and skills to serve their clients in an honest and ethical manner, and can increase public confidence and trust that members will act ethically.

### LOS 1.c

A profession is an occupational group that has requirements of specialized expert knowledge. Professions establish trust by requiring high standards of expertise, setting standards for ethical behavior, and monitoring professional conduct.

### LOS 1.d

Investment professionals have a special responsibility to use their specialized knowledge and skills to both protect and grow client assets. The fact that investment management is an intangible product makes high ethical standards all the more important in the financial services profession.

### LOS 1.e

Some financial professionals are held to a suitability standard, while others are held to a fiduciary standard. Suitability refers to the match between client return requirements and risk tolerances and the characteristics of the securities recommended. A fiduciary standard requires professionals to act in the best interests of the client.

### LOS 1.f

Challenges to ethical behavior include overestimating one’s own ethical character, considering only near-term consequences and not longer-term consequences of behavior, and letting situational (external) influences, such as peer pressure, unduly affect one’s decisions and behavior.

### LOS 1.g

Not all unethical actions are illegal, and not all illegal actions are unethical. Laws are more specific than ethical principles and often address prior unethical behavior. Ethical behavior requires more judgment; acts such as civil disobedience may be considered ethical even when they are illegal.

### LOS 1.h

A framework for ethical decision making is designed to lead to better decisions by identifying the stakeholders affected and the conflicts of interest among them, considering alternative actions and the relevant situational influences on decision makers, seeking out different perspectives, and evaluating decisions to see if they had unintended consequences.

# ANSWER KEY FOR MODULE QUIZ

## Module Quiz 1.1

1. **C** A code of ethics specifies a minimum level of acceptable conduct for a group or organization, whereas "best practices" are suggested behavior, not a minimum acceptable level. (LOS 1.b)

2. **A** A professional code of conduct communicates to the public that members have promised to uphold a minimum level of ethical conduct when acting for clients. This is no guarantee that all members will follow the code at all times. A code of conduct may include specific standards of behavior or only state principles of conduct without specific standards or guidance. (LOS 1.b)

3. **C** Situational factors are those external to the decision makers, such as financial rewards and desire to please coworkers or others. Researchers have found that external factors are often more likely than a lack of personal ethics to lead to poor ethical decisions. (LOS 1.f)

4. **B** A code of ethics is considered a higher standard of behavior as it goes beyond simply legality of behavior. Compliance with the ethical principles of a code of ethics often requires judgment in balancing the interests of various stakeholders and consideration of short-term effects with longer-term effects of decisions. Some behavior that is illegal, such as civil disobedience or "whistle-blowing," is considered to be ethical behavior by many. (LOS 1.g)

5. **C** A decision-making framework that includes the ethical aspects of the decision should consider the conflicts among the interests of various stakeholders so that decision makers can use the company's stated ethical principles and their judgment to balance these interests in an ethical manner. Profit maximization, at least in the short term, does not necessarily follow from sound ethical judgment. While integrating ethics into the decision-making process can consider and reduce unintended ethical consequences of a decision, avoiding them altogether can never be assured. (LOS 1.h)

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1. "Ethics and Trust in the Investment Profession," Bidhan L Parmar, PhD, Dorothy C. Kelly, CFA, and David B. Stevens, CFA, in CFA Program 2020 Level I Curriculum, Volume 1 (CFA Institute, 2019).

The following is a review of the Ethical and Professional Standards principles designed to address the learning outcome statements set forth by CFA Institute. Cross-Reference to CFA Institute Assigned Reading #2.

# READING 2: CODE OF ETHICS AND STANDARDS OF PROFESSIONAL CONDUCT

**Study Session 1**

## EXAM FOCUS

In addition to reading this review of the ethics material, we strongly recommend that all candidates for the CFA® examination read the *Standards of Practice Handbook 11th Edition* (2014) multiple times. As a Level I CFA candidate, it is your responsibility to comply with the *Code and Standards*. The complete *Code and Standards* are reprinted in Volume 1 of the CFA Program Curriculum.

## MODULE 2.1: CODE AND STANDARDS

Video covering this content is available online.

**LOS 2.a: Describe the structure of the CFA Institute Professional Conduct Program and the process for the enforcement of the Code and Standards.**

*CFA® Program Curriculum, Volume 1, page 39*

The CFA Institute Professional Conduct Program is covered by the CFA Institute Bylaws and the Rules of Procedure for Proceedings Related to Professional Conduct. The Program is based on the principles of fairness of the process to members and candidates and maintaining the confidentiality of the proceedings. The CFA Institute Board of Governors has overall responsibility for the Professional Conduct Program and its Disciplinary Review Committee is responsible for enforcing of the Code and Standards.

The CFA Institute Professional Conduct staff conducts inquiries related to professional conduct. Several circumstances can prompt such an inquiry:

1. Self-disclosure by members or candidates on their annual Professional Conduct Statements of involvement in civil litigation or a criminal investigation, or that the member or candidate is the subject of a written complaint.
2. Written complaints about a member or candidate's professional conduct that are received by the Professional Conduct staff.
3. Evidence of misconduct by a member or candidate that the Professional Conduct staff received through public sources, such as a media article or broadcast.
4. A report by a CFA exam proctor of a possible violation during the examination.
5. Analysis of exam materials and monitoring of social media by CFA Institute.

Once an inquiry has begun, the Professional Conduct staff may request (in writing) an explanation from the subject member or candidate and may: (1) interview the subject member

or candidate, (2) interview the complainant or other third parties, and/or (3) collect documents and records relevant to the investigation.

The Professional Conduct staff may decide: (1) that no disciplinary sanctions are appropriate, (2) to issue a cautionary letter, or (3) to discipline the member or candidate. In a case where the Professional Conduct staff finds a violation has occurred and proposes a disciplinary sanction, the member or candidate may accept or reject the sanction. If the member or candidate chooses to reject the sanction, the matter will be referred to a disciplinary review panel of CFA Institute members for a hearing. Sanctions imposed may include condemnation by the member's peers or suspension of candidate's continued participation in the CFA Program.

**LOS 2.b: State the six components of the Code of Ethics and the seven Standards of Professional Conduct.**

*CFA® Program Curriculum, Volume 1, page 46*

## Code of Ethics

Members of CFA Institute [including Chartered Financial Analyst® (CFA®) charterholders] and candidates for the CFA designation ("Members and Candidates") must:[1]

- Act with integrity, competence, diligence, respect, and in an ethical manner with the public, clients, prospective clients, employers, employees, colleagues in the investment profession, and other participants in the global capital markets.
- Place the integrity of the investment profession and the interests of clients above their own personal interests.
- Use reasonable care and exercise independent professional judgment when conducting investment analysis, making investment recommendations, taking investment actions, and engaging in other professional activities.
- Practice and encourage others to practice in a professional and ethical manner that will reflect credit on themselves and the profession.
- Promote the integrity and viability of the global capital markets for the ultimate benefit of society.
- Maintain and improve their professional competence and strive to maintain and improve the competence of other investment professionals.

## The Standards of Professional Conduct

I. Professionalism

II. Integrity of Capital Markets

III. Duties to Clients

IV. Duties to Employers

V. Investment Analysis, Recommendations, and Actions

VI. Conflicts of Interest

VII. Responsibilities as a CFA Institute Member or CFA Candidate

**LOS 2.c: Explain the ethical responsibilities required by the Code and Standards, including the sub-sections of each Standard.**

*CFA® Program Curriculum, Volume 1, page 46*

## Standards of Professional Conduct[2]

**I. PROFESSIONALISM**

A. **Knowledge of the Law.** Members and Candidates must understand and comply with all applicable laws, rules, and regulations (including the CFA Institute *Code of Ethics* and *Standards of Professional Conduct*) of any government, regulatory organization, licensing agency, or professional association governing their professional activities. In the event of conflict, Members and Candidates must comply with the more strict law, rule, or regulation. Members and Candidates must not knowingly participate or assist in any violation of laws, rules, or regulations and must dissociate themselves from any such violation.

B. **Independence and Objectivity.** Members and Candidates must use reasonable care and judgment to achieve and maintain independence and objectivity in their professional activities. Members and Candidates must not offer, solicit, or accept any gift, benefit, compensation, or consideration that reasonably could be expected to compromise their own or another's independence and objectivity.

C. **Misrepresentation.** Members and Candidates must not knowingly make any misrepresentations relating to investment analysis, recommendations, actions, or other professional activities.

D. **Misconduct.** Members and Candidates must not engage in any professional conduct involving dishonesty, fraud, or deceit or commit any act that reflects adversely on their professional reputation, integrity, or competence.

**II. INTEGRITY OF CAPITAL MARKETS**

A. **Material Nonpublic Information.** Members and Candidates who possess material nonpublic information that could affect the value of an investment must not act or cause others to act on the information.

B. **Market Manipulation.** Members and Candidates must not engage in practices that distort prices or artificially inflate trading volume with the intent to mislead market participants.

**III. DUTIES TO CLIENTS**

A. **Loyalty, Prudence, and Care.** Members and Candidates have a duty of loyalty to their clients and must act with reasonable care and exercise prudent judgment. Members and Candidates must act for the benefit of their clients and place their clients' interests before their employer's or their own interests.

B. **Fair Dealing.** Members and Candidates must deal fairly and objectively with all clients when providing investment analysis, making investment recommendations, taking investment action, or engaging in other professional activities.

C. **Suitability.**

1. When Members and Candidates are in an advisory relationship with a client, they must:
    a. Make a reasonable inquiry into a client's or prospective clients' investment experience, risk and return objectives, and financial constraints prior to making any investment recommendation or taking investment action and must reassess and update this information regularly.
    b. Determine that an investment is suitable to the client's financial situation and consistent with the client's written objectives, mandates, and constraints before making an investment recommendation or taking investment action.
    c. Judge the suitability of investments in the context of the client's total portfolio.
2. When Members and Candidates are responsible for managing a portfolio to a specific mandate, strategy, or style, they must make only investment recommendations or take only investment actions that are consistent with the stated objectives and constraints of the portfolio.

D. **Performance Presentation.** When communicating investment performance information, Members or Candidates must make reasonable efforts to ensure that it is fair, accurate, and complete.

E. **Preservation of Confidentiality.** Members and Candidates must keep information about current, former, and prospective clients confidential unless:
    1. The information concerns illegal activities on the part of the client or prospective client,
    2. Disclosure is required by law, or
    3. The client or prospective client permits disclosure of the information.

## IV. DUTIES TO EMPLOYERS

A. **Loyalty.** In matters related to their employment, Members and Candidates must act for the benefit of their employer and not deprive their employer of the advantage of their skills and abilities, divulge confidential information, or otherwise cause harm to their employer.

B. **Additional Compensation Arrangements.** Members and Candidates must not accept gifts, benefits, compensation, or consideration that competes with or might reasonably be expected to create a conflict of interest with their employer's interest unless they obtain written consent from all parties involved.

C. **Responsibilities of Supervisors.** Members and Candidates must make reasonable efforts to ensure that anyone subject to their supervision or authority complies with applicable laws, rules, regulations, and the Code and Standards.

## V. INVESTMENT ANALYSIS, RECOMMENDATIONS, AND ACTIONS

A. **Diligence and Reasonable Basis.** Members and Candidates must:

1. Exercise diligence, independence, and thoroughness in analyzing investments, making investment recommendations, and taking investment actions.
2. Have a reasonable and adequate basis, supported by appropriate research and investigation, for any investment analysis, recommendation, or action.

B. **Communication with Clients and Prospective Clients.** Members and Candidates must:

1. Disclose to clients and prospective clients the basic format and general principles of the investment processes they use to analyze investments, select securities, and construct portfolios and must promptly disclose any changes that might materially affect those processes.
2. Disclose to clients and prospective clients significant limitations and risks associated with the investment process.
3. Use reasonable judgment in identifying which factors are important to their investment analyses, recommendations, or actions and include those factors in communications with clients and prospective clients.
4. Distinguish between fact and opinion in the presentation of investment analysis and recommendations.

C. **Record Retention.** Members and Candidates must develop and maintain appropriate records to support their investment analysis, recommendations, actions, and other investment-related communications with clients and prospective clients.

## VI. CONFLICTS OF INTEREST

A. **Disclosure of Conflicts.** Members and Candidates must make full and fair disclosure of all matters that could reasonably be expected to impair their independence and objectivity or interfere with respective duties to their clients, prospective clients, and employer. Members and Candidates must ensure that such disclosures are prominent, are delivered in plain language, and communicate the relevant information effectively.

B. **Priority of Transactions.** Investment transactions for clients and employers must have priority over investment transactions in which a Member or Candidate is the beneficial owner.

C. **Referral Fees.** Members and Candidates must disclose to their employer, clients, and prospective clients, as appropriate, any compensation, consideration, or benefit received by, or paid to, others for the recommendation of products or services.

## VII. RESPONSIBILITIES AS A CFA INSTITUTE MEMBER OR CFA CANDIDATE

A. **Conduct as Participants in CFA Institute Programs.** Members and Candidates must not engage in any conduct that compromises the reputation or integrity of CFA Institute or the CFA designation or the integrity, validity, or security of CFA Institute programs.

B. **Reference to CFA Institute, the CFA Designation, and the CFA Program**. When referring to CFA Institute, CFA Institute membership, the CFA designation, or candidacy in the CFA Program, Members and Candidates must not misrepresent or exaggerate the meaning or implications of membership in CFA Institute, holding the CFA designation, or candidacy in the CFA Program.

## MODULE QUIZ 2.1

*To best evaluate your performance, enter your quiz answers online.*

1. In the case of a complaint about a member's professional conduct, CFA Institute Professional Conduct Program staff are *least likely* to:
   A. review documents and records related to the complaint.
   B. request an interview with the member or with the party making the complaint.
   C. suspend the member's right to use the CFA designation while an investigation is in progress.
2. Which of the following requirements for members and candidates is one of the six components of the Code of Ethics?
   A. Maintain and improve their professional competence.
   B. Do not act or cause others to act on material nonpublic information.
   C. Distinguish between fact and opinion when presenting investment analysis.
3. If a member or candidate is offered an additional compensation arrangement by a client, which of the seven Standards of Professional Conduct states the requirements the member or candidate must follow?
   A. Duties to Clients.
   B. Conflicts of Interest.
   C. Duties to Employers.

# ANSWER KEY FOR MODULE QUIZ

## Module Quiz 2.1

1. **C** The process for enforcing the Code and Standards does not include suspending a member or candidate while an inquiry is in progress. If CFA Institute Professional Conduct staff receive information that prompts an inquiry, the staff may request information from the member or candidate, interview parties who initiated a complaint, or review relevant records and documents. (LOS 2.a)

2. **A** One of the six components of the Code of Ethics requires members and candidates to "maintain and improve their professional competence and strive to maintain and improve the competence of other investment professionals." The other two answer choices are required by the Standards of Professional Conduct but are not components of the Code of Ethics. (LOS 2.b)

3. **C** The standard related to additional compensation arrangements is a subsection of Standard IV Duties to Employers. (LOS 2.c)

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1. Copyright 2014, CFA Institute. Reproduced and republished from "The Code of Ethics," from *Standards of Practice Handbook, 11th Ed.*, 2014, with permission from CFA Institute. All rights reserved.

2. Ibid.

The following is a review of the Ethical and Professional Standards principles designed to address the learning outcome statements set forth by CFA Institute. Cross-Reference to CFA Institute Assigned Reading #3.

# READING 3: GUIDANCE FOR STANDARDS I–VII

**Study Session 1**

## EXAM FOCUS

The Standards of Professional Conduct comprise seven Standards (I–VII) and a total of 22 subsections. These Standards and their application are described in the *Standards of Practice Handbook, 11th Edition, 2014,* published by CFA Institute. We recommend carefully reading the *Standards of Practice Handbook* multiple times in preparation for your Level I exam (yes, the whole thing, including all examples). Fifteen percent of your exam questions will be based on this book and the two relatively short readings concerning Global Investment Performance Standards (GIPS). Given that much of this material must simply be memorized, we also suggest that your final reading of the *Standards of Practice Handbook* be on the Friday prior to your exam. You probably don't need to read all the examples that day, but if you highlighted some points during an earlier reading, you can revisit those as you go through all the Standards, the guidance, and the recommended best practices.

## MODULE 3.1: GUIDANCE FOR STANDARDS I(A) AND I(B)

Video covering this content is available online.

**LOS 3.a: Demonstrate the application of the Code of Ethics and Standards of Professional Conduct to situations involving issues of professional integrity.**

**LOS 3.b: Distinguish between conduct that conforms to the Code and Standards and conduct that violates the Code and Standards.**

**LOS 3.c: Recommend practices and procedures designed to prevent violations of the Code of Ethics and Standards of Professional Conduct.**

*CFA® Program Curriculum, Volume 1, page 55*

The *Standards of Practice Handbook* is included in its entirety in the CFA curriculum as Readings 2 and 3 of Volume 1. You can also download a PDF of the *Standards of Practice Handbook* at the website for CFA Institute, www.cfainstitute.org. A third alternative is to purchase the *Standards of Practice Handbook* through Amazon (make sure you get the 11th edition) for about $30 or get the Kindle edition for $0.99.

In our summary of the Standards of Practice, we focus on describing three things: (1) actions that clearly violate the subsection, (2) the behaviors that each subsection is intended to either encourage or discourage, and (3) recommended best practices for members and their firms.

In many cases the actions that members and candidates must not take are explained using terms open to interpretation, such as “reasonable,” “adequate,” and “token.”

Some examples from the Standards themselves are:

…use reasonable care and judgment to achieve…

…accept any gift, that reasonably could be expected to compromise…

…act with reasonable care and exercise prudent judgment…

…deal fairly and objectively with all clients…

...make a reasonable inquiry into…

…make reasonable efforts to ensure…

…might reasonably be expected to create a conflict of interest with…

…Have a reasonable and adequate basis…

…Use reasonable judgment in…

…matters that could be reasonably expected to impair…

The requirement of the LOS is that you know what constitutes a violation, not that you draw a distinction between what is “reasonable” and what is not in a given situation. We believe the exam writers take this into account and that if they intend, for example, to test whether a recommendation has been given without reasonable care and judgment, it will likely be clear either that the care and judgment exhibited by the analyst did not rise to the level of “reasonable,” or that it did.

No monetary value for a “token” gift is given in the Standards, although it is recommended that a firm establish such a monetary value for its employees. Here, again, the correct answer to a question will not likely hinge on candidate’s determination of what is a token gift and what is not. Questions should be clear in this regard. A business dinner is likely a token gift, but a week at a condominium in Aspen or tickets to the Super Bowl are likely not. Always look for clues in the questions that lead you to the question-writer’s preferred answer choice, such as “lavish” entertainment and “luxury” accommodations.

Next, we present a summary of each subsection of the Standards of Professional Conduct. For each one, we first detail actions that violate the Standard and then list actions and behaviors that are recommended within the Standards. We suggest you learn the violations especially well so you understand that the other items are recommended. For the exam, it is not necessary to memorize the Standard number and subsection letter. Knowing that an action violates, for example, Professionalism, rather than Duties to Employers or Duties to Clients, should be sufficient in this regard. Note that some actions may violate more than one Standard.

One way to write questions for this material is to offer a reason that might make one believe a Standard does not apply in a particular situation. In most, if not all, cases the “reason” does not change the requirement of the Standard. If you are prohibited from some action, the motivations for the action or other circumstances simply do not matter. If the Standard says it’s a violation, it’s a violation. An exception is when intent is key to the Standard, such as intending to mislead clients or market participants in general.

## STANDARD I: PROFESSIONALISM[1]

## Standard I(A) Knowledge of the Law

Members and Candidates must understand and comply with all applicable laws, rules, and regulations (including the CFA Institute Code of Ethics and Standards of Professional Conduct) of any government, regulatory organization, licensing agency, or professional association governing their professional activities. In the event of conflict, Members and Candidates must comply with the more strict law, rule, or regulation. Members and Candidates must not knowingly participate or assist in and must dissociate from any violation of such laws, rules, or regulations.

The Standards begin with a straightforward statement: Don't violate any laws, rules, or regulations that apply to your professional activities. This includes the Code and Standards, so any violation of the Code and Standards will also violate this subsection.

A member may be governed by different rules and regulations among the Standards, the country in which the member resides, and the country where the member is doing business. Follow the most strict of these, or, put another way, do not violate any of the three sets of rules and regulations.

If you know that violations of applicable rules or laws are taking place, either by coworkers or clients, CFA Institute strongly encourages members and candidates to report potential violations.[2] One way to do so is to approach your supervisor or compliance department to remedy the situation. If they will not or cannot, then you must dissociate from the activity (e.g., not working with a trading group you know is not allocating client trades properly according to the Standard on Fair Dealing, or not using marketing materials that you know or should know are misleading or erroneous). If this cannot be accomplished, you may, in an extreme case, have to resign from the firm to be in compliance with this Standard.

## Recommendations for Members

- Establish, or encourage employer to establish, procedures to keep employees informed of changes in relevant laws, rules, and regulations.
- Review, or encourage employer to review, the firm's written compliance procedures on a regular basis.
- Maintain, or encourage employer to maintain, copies of current laws, rules, and regulations.
- When in doubt about legality, consult supervisor, compliance personnel, or a lawyer.
- When dissociating from violations, keep records documenting the violations, encourage employer to bring an end to the violations.
- There is no requirement in the Standards to report wrongdoers, but local law may require it; members are "strongly encouraged" to report violations to CFA Institute Professional Conduct Program.

## Recommendations for Firms

- Have a code of ethics.
- Provide employees with information on laws, rules, and regulations governing professional activities.
- Have procedures for reporting suspected violations.

## Standard I(B) Independence and Objectivity

Members and Candidates must use reasonable care and judgment to achieve and maintain independence and objectivity in their professional activities. Members and Candidates must not offer, solicit, or accept any gift, benefit, compensation, or consideration that reasonably could be expected to compromise their own or another's independence and objectivity.

Analysts may face pressure or receive inducements to give a security a specific rating, to select certain outside managers or vendors, or to produce favorable or unfavorable research and conclusions. Members who allow their investment recommendations or analysis to be influenced by such pressure or inducements will have violated the requirement to use reasonable care and to maintain independence and objectivity in their professional activities. Allocating shares in oversubscribed IPOs to personal accounts is a violation.

Normal business entertainment is permitted. Members who accept, solicit, or offer things of value that could be expected to influence the member's or others' independence or objectivity are violating the Standard. Gifts from clients are considered less likely to compromise independence and objectivity than gifts from other parties. Client gifts must be disclosed to the member's employer prior to acceptance, if possible, but after acceptance, if not.

Members may prepare reports paid for by the subject firm if compensation is a flat rate not tied to the conclusions of the report (and if the fact that the research is issuer-paid is disclosed). Accepting compensation that is dependent on the conclusions, recommendations, or market impact of the report, and failure to disclose that research is issuer-paid, are violations of this Standard.

### Recommendations for Members

Members or their firms should pay for their own travel to company events or tours when practicable and limit use of corporate aircraft to trips for which commercial travel is not an alternative.

### Recommendations for Firms

- Restrict employee participation in IPOs and private placements, require pre-approval for participation.
- Appoint a compliance officer, have written policies on independence and objectivity and clear procedures for reporting violations.
- Limit gifts, other than from clients, to token items only.

# MODULE 3.2: GUIDANCE FOR STANDARDS I(C) AND I(D)

Video covering this content is available online.

## Standard I(C) Misrepresentation

Members and Candidates must not knowingly make any misrepresentations relating to investment analysis, recommendations, actions, or other professional activities.

Misrepresentation includes knowingly misleading investors, omitting relevant information, presenting selective data to mislead investors, and plagiarism. Plagiarism is using reports, forecasts, models, ideas, charts, graphs, or spreadsheets created by others without crediting

the source. Crediting the source is not required when using projections, statistics, and tables from recognized financial and statistical reporting services. When using models developed or research done by other members of the firm, it is permitted to omit the names of those who are no longer with the firm as long as the member does not represent work previously done by others as his alone.

Actions that would violate the Standard include:

- Presenting third-party research as your own, without attribution to the source.
- Guaranteeing a specific return on securities that do not have an explicit guarantee from a government body or financial institution.
- Selecting a valuation service because it puts the highest value on untraded security holdings.
- Selecting a performance benchmark that is not comparable to the investment strategy employed.
- Presenting performance data or attribution analysis that omits accounts or relevant variables.
- Offering false or misleading information about the analyst's or firm's capabilities, expertise, or experience.
- Using marketing materials from a third party (outside advisor) that are misleading.

## Recommendations for Members

- Prepare a summary of experience, qualifications, and services a member is able to perform.
- Encourage employers to develop procedures for verifying marketing materials provided by third parties concerning their capabilities, products, and services.
- Cite the source of any summaries of materials provided by others.
- Keep copies of all reports, articles, or other materials used in the preparation of research reports.
- Provide a list, in writing, of the firm's available services and qualifications.
- Periodically review documents and communications of members for any misrepresentation of employee or firm qualifications and capabilities.

# Standard I(D) Misconduct

Members and Candidates must not engage in any professional conduct involving dishonesty, fraud, or deceit or commit any act that reflects adversely on their professional reputation, integrity, or competence.

The first part here regarding professional conduct is clear: no dishonesty, fraud, or deceit. The second part, while it applies to all conduct by the member, specifically requires that the act, "reflects adversely on their professional reputation, integrity, or competence." The guidance states, in fact, that members must not try to use enforcement of this Standard against another member to settle personal, political, or other disputes that are not related to professional ethics or competence.

## Recommendations for Firms

- Develop and adopt a code of ethics and make clear that unethical behavior will not be tolerated.
- Give employees a list of potential violations and sanctions, including dismissal.
- Check references of potential employees.

# MODULE 3.3: GUIDANCE FOR STANDARD II

Video covering this content is available online.

## STANDARD II: INTEGRITY OF CAPITAL MARKETS

### Standard II(A) Material Nonpublic Information

Members and Candidates who possess material nonpublic information that could affect the value of an investment must not act or cause others to act on the information.

Information is "material" if its disclosure would affect the price of a security or if a reasonable investor would want the information before making an investment decision. Information that is ambiguous as to its likely effect on price may not be considered material.

Information is "nonpublic" until it has been made available to the marketplace. An analyst conference call is not public disclosure. Selective disclosure of information by corporations creates the potential for insider-trading violations.

The prohibition against acting on material nonpublic information extends to mutual funds containing the subject securities as well as related swaps and options contracts. It is the member's responsibility to determine if information she receives has been publicly disseminated prior acting or causing others to act on it.

Some members and candidates may be involved in transactions during which they are provided with material nonpublic information by firms (e.g., investment banking transactions). Members and candidates may use this information for its intended purpose, but must not use the information for any other purpose unless it becomes public information.

Under the so-called **mosaic theory**, reaching an investment conclusion through perceptive analysis of public information combined with non-material nonpublic information is not a violation of the Standard.

### Recommendations for Members

- Make reasonable efforts to achieve public dissemination by the firm of information they possess.
- Encourage their firms to adopt procedures to prevent the misuse of material nonpublic information.

### Recommendations for Firms

Use a firewall within the firm, with elements including:

- Exercise substantial control of relevant interdepartmental communications through a clearance area, such as the compliance or legal department.
- Review employee trades.

- Maintain “watch,” “restricted,” and “rumor” lists.

Monitor and restrict proprietary trading while a firm is in possession of material nonpublic information. However, prohibiting all proprietary trading while a firm is in possession of material nonpublic information may be inappropriate because it may send a signal to the market. In these cases, firms should only take the opposite side of unsolicited customer trades.

## Standard II(B) Market Manipulation

Members and Candidates must not engage in practices that distort prices or artificially inflate trading volume with the intent to mislead market participants.

Member actions may affect security values and trading volumes without violating this Standard. The key point here is that if there is the *intent to mislead*, then the Standard is violated. Of course, spreading false information to affect prices or volume is a violation of this Standard as is making trades intended to mislead market participants.

### MODULE QUIZ 3.1, 3.2, 3.3

*To best evaluate your performance, enter your quiz answers online.*

1. In situations where the laws of a member or candidate’s country of residence, the local laws of regions where the member or candidate does business, and the Code and Standards specify different requirements, the member or candidate must abide by:
   A. local law or the Code and Standards, whichever is stricter.
   B. the Code and Standards or his country’s laws, whichever are stricter.
   C. the strictest of local law, his country’s laws, or the Code and Standards.
2. According to the Standard on independence and objectivity, members and candidates:
   A. may accept gifts or bonuses from clients.
   B. may not accept compensation from an issuer of securities in return for producing research on those securities.
   C. should consider credit ratings issued by recognized agencies to be objective measures of credit quality.
3. Bill Cooper finds a table of historical bond yields on the website of the U.S. Treasury that supports the work he has done in his analysis and includes the table as part of his report without citing the source. Has Cooper violated the Code and Standards?
   A. Yes, because he did not cite the source of the table.
   B. Yes, because he did not verify the accuracy of the information.
   C. No, because the table is from a recognized source of financial or statistical data.
4. Which of the following statements about the Standard on misconduct is *most accurate*?
   A. Misconduct applies only to a member or candidate’s professional activities.
   B. Neglecting to perform due diligence when required is an example of misconduct.
   C. A member or candidate commits misconduct by engaging in any illegal activity.
5. Ed Ingus, CFA, visits the headquarters and main plant of Bullitt Company and observes that inventories of unsold goods appear unusually large. From the CFO, he learns that a recent increase in returned items may result in earnings for the current quarter that are below analysts’ estimates. Based on his visit, Ingus changes his recommendation on Bullitt to “Sell.” Has Ingus violated the Standard concerning material nonpublic information?
   A. Yes.
   B. No, because the information he used is not material.
   C. No, because his actions are consistent with the mosaic theory.
6. Green Brothers, an emerging market fund manager, has two of its subsidiaries simultaneously buy and sell emerging market stocks. In its marketing literature, Green Brothers cites the overall emerging market volume as evidence of the market’s liquidity. As a result of its actions, more investors participate in the emerging markets fund. Green Brothers

*most likely*:

A. did not violate the Code and Standards.
B. violated the Standard regarding market manipulation.
C. violated the Standard regarding performance presentation.

# MODULE 3.4: GUIDANCE FOR STANDARDS III(A) AND III(B)

Video covering this content is available online.

## STANDARD III: DUTIES TO CLIENTS

### Standard III(A) Loyalty, Prudence, and Care

Members and Candidates have a duty of loyalty to their clients and must act with reasonable care and exercise prudent judgment. Members and Candidates must act for the benefit of their clients and place their clients' interests before their employer's or their own interests.

Client interests always come first. Although this Standard does not impose a fiduciary duty on members or candidates where one did not already exist, it does require members and candidates to act in their clients' best interests and recommend products that are suitable given their clients' investment objectives and risk tolerances. Members and candidates must:

- Exercise the prudence, care, skill, and diligence under the circumstances that a person acting in a like capacity and familiar with such matters would use.
- Manage pools of client assets in accordance with the terms of the governing documents, such as trust documents or investment management agreements.
- Make investment decisions in the context of the total portfolio.
- Inform clients of any limitations in an advisory relationship (e.g., an advisor who may only recommend her own firm's products).
- Vote proxies in an informed and responsible manner. Due to cost-benefit considerations, it may not be necessary to vote all proxies.
- Client brokerage, or "soft dollars" or "soft commissions," must be used to benefit the client.
- The "client" may be the investing public as a whole rather than a specific entity or person.

### Recommendations for Members

Submit to clients, at least quarterly, itemized statements showing all securities in custody and all debits, credits, and transactions.

Encourage firms to address these topics when drafting policies and procedures regarding fiduciary duty:

- Follow applicable rules and laws.
- Establish investment objectives of client.
- Consider suitability of a portfolio relative to the client's needs and circumstances, the investment's basic characteristics, or the basic characteristics of the total portfolio.
- Diversify.
- Deal fairly with all clients in regard to investment actions.

- Disclose conflicts.
- Disclose compensation arrangements.
- Vote proxies in the best interest of clients and ultimate beneficiaries.
- Maintain confidentiality.
- Seek best execution.

## Standard III(B) Fair Dealing

Members and Candidates must deal fairly and objectively with all clients when providing investment analysis, making investment recommendations, taking investment action, or engaging in other professional activities.

Do not discriminate against any clients when disseminating recommendations or taking investment action. “Fairly” does not mean “equally.” In the normal course of business, there will be differences in the time emails, faxes, and other communications are received by different clients.

Different service levels are acceptable, but they must not negatively affect or disadvantage any clients. Disclose the different service levels to all clients and prospects, and make premium levels of service available to all those willing to pay for them.

Give all clients a fair opportunity to act on every recommendation. Clients who are unaware of a change in the recommendation for a security should be advised of the change before an order for the security is accepted.

Treat clients fairly in light of their investment objectives and circumstances. Treat both individual and institutional clients in a fair and impartial manner. Members and candidates should not take advantage of their position in the industry to disadvantage clients (e.g., taking shares of an oversubscribed IPO).

## Recommendations for Members

- Encourage firms to establish compliance procedures requiring proper dissemination of investment recommendations and fair treatment of all customers and clients.
- Maintain a list of clients and holdings—use to ensure that all holders are treated fairly.

## Recommendations for Firms

- Limit the number of people who are aware that a change in recommendation will be made.
- Shorten the time frame between decision and dissemination.
- Publish personnel guidelines for pre-dissemination—have in place guidelines prohibiting personnel who have prior knowledge of a recommendation from discussing it or taking action on the pending recommendation.
- Disseminate new or changed recommendations simultaneously to all clients who have expressed an interest or for whom an investment is suitable.
- Develop written trade allocation procedures—ensure fairness to clients, timely and efficient order execution, and accuracy of client positions.
- Disclose trade allocation procedures.

- Establish systematic account review—ensure that no client is given preferred treatment and that investment actions are consistent with the account's objectives.
- Disclose available levels of service.

# MODULE 3.5: GUIDANCE FOR STANDARDS III(C), III(D), AND III(E)

Video covering this content is available online.

## Standard III(C) Suitability

1. When Members and Candidates are in an advisory relationship with a client, they must:
    a. Make a reasonable inquiry into a client's or prospective client's investment experience, risk and return objectives, and financial constraints prior to making any investment recommendation or taking investment action and must reassess and update this information regularly.
    b. Determine that an investment is suitable to the client's financial situation and consistent with the client's written objectives, mandates, and constraints before making an investment recommendation or taking investment action.
    c. Judge the suitability of investments in the context of the client's total portfolio.
2. When Members and Candidates are responsible for managing a portfolio to a specific mandate, strategy, or style, they must make only investment recommendations or take only investment actions that are consistent with the stated objectives and constraints of the portfolio.

In advisory relationships, members must gather client information at the beginning of the relationship, in the form of an investment policy statement (IPS). Consider clients' needs and circumstances and, thus, their risk tolerance. Consider whether or not the use of leverage is suitable for the client.

If a member is responsible for managing a fund to an index or other stated mandate, he must select only investments that are consistent with the stated mandate.

## Unsolicited Trade Requests

An investment manager may receive a client request to purchase a security that the manager knows is unsuitable, given the client's investment policy statement. The trade may or may not have a material effect on the risk characteristics of the client's total portfolio and the requirements are different for each case. In either case, however, the manager should not make the trade until he has discussed with the client the reasons (based on the IPS) that the trade is unsuitable for the client's account.

If the manager determines that the *effect on the risk/return profile of the client's total portfolio is minimal*, the manager, after discussing with the client how the trade does not fit the IPS goals and constraints, may follow his firm's policy with regard to unsuitable trades. Regardless of firm policy, the client must acknowledge the discussion and an understanding of why the trade is unsuitable.

If the trade would have a *material impact on the risk/return profile of the client's total portfolio*, one option is to update the IPS so the client accepts a changed risk profile that would permit the trade. If the client will not accept a changed IPS, the manager may follow

firm policy, which may allow the trade to be made in a separate client-directed account. In the absence of other options, the manager may need to reconsider whether to maintain the relationship with the client.

## Recommendations for Members

- For each client, put the needs, circumstances, and investment objectives into a written IPS.
- Consider the type of client and whether there are separate beneficiaries, investor objectives (return and risk), investor constraints (liquidity needs, expected cash flows, time, tax, and regulatory and legal circumstances), and performance measurement benchmarks.
- Review the investor's objectives and constraints periodically to reflect any changes in client circumstances.

## Standard III(D) Performance Presentation

When communicating investment performance information, Members and Candidates must make reasonable efforts to ensure that it is fair, accurate, and complete.

Members must not misstate performance or mislead clients or prospects about their investment performance or their firm's investment performance.

Members must not misrepresent past performance or reasonably expected performance, and must not state or imply the ability to achieve a rate of return similar to that achieved in the past.

For brief presentations, members must make detailed information available on request and indicate that the presentation has offered only limited information.

## Recommendations for Members

- Encourage firms to adhere to Global Investment Performance Standards.
- Consider the sophistication of the audience to whom a performance presentation is addressed.
- Present the performance of a weighted composite of similar portfolios rather than the performance of a single account.
- Include terminated accounts as part of historical performance and clearly state when they were terminated.
- Include all appropriate disclosures to fully explain results (e.g., model results included, gross or net of fees, etc.).
- Maintain data and records used to calculate the performance being presented.

## Standard III(E) Preservation of Confidentiality

Members and Candidates must keep information about current, former, and prospective clients confidential unless:

1. The information concerns illegal activities on the part of the client;
2. Disclosure is required by law; or

3. The client or prospective client permits disclosure of the information.

If illegal activities by a client are involved, members may have an obligation to report the activities to authorities.

The confidentiality Standard extends to former clients as well.

The requirements of this Standard are not intended to prevent members and candidates from cooperating with a CFA Institute Professional Conduct Program (PCP) investigation.

## Recommendations for Members

- Members should avoid disclosing information received from a client except to authorized coworkers who are also working for the client.
- Members should follow firm procedures for storage of electronic data and recommend adoption of such procedures if they are not in place.

### MODULE QUIZ 3.4, 3.5

*To best evaluate your performance, enter your quiz answers online.*

1. Cobb, Inc., has hired Jude Kasten, CFA, to manage its pension fund. The client(s) to whom Kasten owes a duty of loyalty are:
   A. Cobb's management.
   B. the shareholders of Cobb, Inc.
   C. the beneficiaries of the pension fund.
2. Which of the following actions is *most likely* a violation of the Standard on fair dealing?
   A. A portfolio manager allocates IPO shares to all client accounts, including her brother's fee-based retirement account.
   B. An investment firm routinely begins trading for its own account immediately after announcing recommendation changes to clients.
   C. After releasing a general recommendation to all clients, an analyst calls the firm's largest institutional clients to discuss the recommendation in more detail.
3. The Standard regarding suitability *most likely* requires that:
   A. an advisor must analyze an investment's suitability for the client prior to recommending or acting on the investment.
   B. a member or candidate must decline to carry out an unsolicited transaction that she believes is unsuitable for the client.
   C. when managing a fund to an index, a manager who is evaluating potential investments must consider their suitability for the fund's shareholders.
4. Which of the following is *most likely* a recommended procedure for complying with the Standard on performance presentation?
   A. Exclude terminated accounts from past performance history.
   B. Present the performance of a representative account to show how a composite has performed.
   C. Consider the level of financial knowledge of the audience to whom the performance is presented.
5. The CFA Institute Professional Conduct Program (PCP) has begun an investigation into Chris Jones, a Level II CFA candidate, and a number of his CFA charterholder colleagues. Jones has access to confidential client records that could be useful in clearing his name and wishes to share this information with the PCP. Which of the following *most accurately* describes Jones's duties with regard to preservation of confidentiality?
   A. Sharing the confidential information with the PCP would violate the Standards.
   B. The Standards encourage, but do not require, that Jones support the PCP investigation into his colleagues.

C. Jones may share confidential information about former clients with the PCP but may not share confidential information about current clients.

# MODULE 3.6: GUIDANCE FOR STANDARD IV

Video covering this content is available online.

## STANDARD IV: DUTIES TO EMPLOYERS

### Standard IV(A) Loyalty

In matters related to their employment, Members and Candidates must act for the benefit of their employer and not deprive their employer of the advantage of their skills and abilities, divulge confidential information, or otherwise cause harm to their employer.

This Standard is applicable to employees. If members are independent contractors, rather than employees, they have a duty to abide by the terms of their agreements.

Members must not engage in any activities that would injure the firm, deprive it of profit, or deprive it of the advantage of employees' skills and abilities.

Members should always place client interests above interests of their employer, but consider the effects of their actions on firm integrity and sustainability.

There is no requirement that the employee put employer interests ahead of family and other personal obligations; it is expected that employers and employees will discuss such matters and balance these obligations with work obligations.

There may be isolated cases where a duty to one's employer may be violated in order to protect clients or the integrity of the market, when the actions are not for personal gain.

Independent practice for compensation is allowed if a notification is provided to the employer fully describing all aspects of the services, including compensation, duration, and the nature of the activities and the employer consents to all terms of the proposed independent practice before it begins.

When leaving an employer, members must continue to act in their employer's best interests until their resignation is effective. Activities that may constitute a violation include:

- Misappropriation of trade secrets.
- Misuse of confidential information.
- Soliciting employer's clients prior to leaving.
- Self-dealing.
- Misappropriation of client lists.

Employer records on any medium (e.g., home computer, tablet, cell phone) are the property of the firm.

When an employee has left a firm, simple knowledge of names and existence of former clients is generally not confidential. There is also no prohibition on the use of experience or knowledge gained while with a former employer. If an agreement exists among employers (e.g., the U.S. "Protocol for Broker Recruiting") that permits brokers to take certain client information when leaving a firm, a member may act within the terms of the agreement without violating the Standard.

Members and candidates must adhere to their employers' policies concerning social media. When planning to leave an employer, members and candidates must ensure that their social media use complies with their employers' policies for notifying clients about employee separations.

## Recommendations for Members

Members are encouraged to give their employer a copy of the Code and Standards.

Best practice is to use separate social media accounts for personal and professional communications.

## Recommendations for Firms

Employers should not have incentive and compensation systems that encourage unethical behavior.

# Standard IV(B) Additional Compensation Arrangements

Members and Candidates must not accept gifts, benefits, compensation, or consideration that competes with or might reasonably be expected to create a conflict of interest with their employer's interest unless they obtain written consent from all parties involved.

Compensation includes direct and indirect compensation from a client and other benefits received from third parties.

Written consent from a member's employer includes email communication.

Understand the difference between an additional compensation arrangement and a gift from a client:

- If a client offers a bonus that depends on the *future performance* of her account, this is an additional compensation arrangement that requires written consent in advance.
- If a client offers a bonus to reward a member for her account's *past performance*, this is a gift that requires disclosure to the member's employer to comply with Standard I(B) Independence and Objectivity.

## Recommendations for Members

Make an immediate written report to the employer detailing any proposed compensation and services, if additional to that provided by the employer.

Members and candidates who are hired to work part time should discuss any arrangements that may compete with their employer's interest at the time they are hired and abide by any limitations their employer identifies.

## Recommendations for Firms

Details of additional compensation, including any performance incentives, should be verified by the offering party.

# Standard IV(C) Responsibilities of Supervisors

Members and Candidates must make reasonable efforts to ensure that anyone subject to their supervision or authority complies with applicable laws, rules, regulations, and the Code and Standards.

Members must make reasonable efforts to prevent employees from violating laws, rules, regulations, or the Code and Standards, as well as make reasonable efforts to detect violations.

An adequate compliance system must meet industry standards, regulatory requirements, and the requirements of the Code and Standards.

Members with supervisory responsibilities have an obligation to bring an inadequate compliance system to the attention of firm's management and recommend corrective action.

A member or candidate faced with no compliance procedures or with procedures he believes are inadequate must decline supervisory responsibility in writing until adequate procedures are adopted by the firm.

If there is a violation, respond promptly and conduct a thorough investigation while increasing supervision or placing limitations on the wrongdoer's activities.

## Recommendations for Members

A member should recommend that his employer adopt a code of ethics. Members should encourage employers to provide their codes of ethics to clients.

Once the compliance program is instituted, the supervisor should:

- Distribute it to the proper personnel.
- Update it as needed.
- Continually educate staff regarding procedures.
- Issue reminders as necessary.
- Require professional conduct evaluations.
- Review employee actions to monitor compliance and identify violations.

## Recommendations for Firms

Employers should not commingle compliance procedures with the firm's code of ethics—this can dilute the goal of reinforcing one's ethical obligations.

While investigating a possible breach of compliance procedures, it is appropriate to limit the suspected employee's activities.

Adequate compliance procedures should:

- Be clearly written.
- Be easy to understand.
- Designate a compliance officer with authority clearly defined.
- Have a system of checks and balances.
- Outline the scope of procedures.
- Outline what conduct is permitted.

- Contain procedures for reporting violations and sanctions.
- Structure incentives so unethical behavior is not rewarded.

# MODULE 3.7: GUIDANCE FOR STANDARD V

Video covering this content is available online.

## STANDARD V: INVESTMENT ANALYSIS, RECOMMENDATIONS, AND ACTIONS

### Standard V(A) Diligence and Reasonable Basis

Members and Candidates must:

1. Exercise diligence, independence, and thoroughness in analyzing investments, making investment recommendations, and taking investment actions.
2. Have a reasonable and adequate basis, supported by appropriate research and investigation, for any investment analysis, recommendation, or action.

The application of this Standard depends on the investment philosophy adhered to, members' and candidates' roles in the investment decision-making process, and the resources and support provided by employers. These factors dictate the degree of diligence, thoroughness of research, and the proper level of investigation required.

The level of research needed to satisfy the requirement for due diligence will differ depending on the product or service offered. A list of things that should be considered prior to making a recommendation or taking investment action includes:

- Global and national economic conditions.
- A firm's financial results and operating history, and the business cycle stage.
- Fees and historical results for a mutual fund.
- Limitations of any quantitative models used.
- A determination of whether peer group comparisons for valuation are appropriate.

### Recommendations for Members

Members should encourage their firms to adopt a policy for periodic review of the quality of third-party research, if they have not. Examples of criteria to use in judging quality are:

- Review assumptions used.
- Determine how rigorous the analysis was.
- Identify how timely the research is.
- Evaluate objectivity and independence of the recommendations.

Members should *encourage their firms to consider* these policies and procedures supporting this Standard:

- Have a policy requiring that research reports and recommendations have a basis that can be substantiated as reasonable and adequate.
- Have detailed, written guidance for proper research and due diligence.

- Have measurable criteria for judging the quality of research, and base analyst compensation on such criteria.
- Have written procedures that provide a minimum acceptable level of scenario testing for computer-based models and include standards for the range of scenarios, model accuracy over time, and a measure of the sensitivity of cash flows to model assumptions and inputs.
- Have a policy for evaluating outside providers of information that addresses the reasonableness and accuracy of the information provided and establishes how often the evaluations should be repeated.
- Adopt a set of standards that provides criteria for evaluating external advisers and states how often a review of external advisers will be performed.

## Standard V(B) Communication With Clients and Prospective Clients

Members and Candidates must:

1. Disclose to clients and prospective clients the basic format and general principles of the investment processes they use to analyze investments, select securities, and construct portfolios and must promptly disclose any changes that might materially affect those processes.
2. Disclose to clients and prospective clients significant limitations and risks associated with the investment process.
3. Use reasonable judgment in identifying which factors are important to their investment analyses, recommendations, or actions and include those factors in communications with clients and prospective clients.
4. Distinguish between fact and opinion in the presentation of investment analyses and recommendations.

All means and types of communication with clients are covered by this Standard, not just research reports or other written communications.

Members must distinguish between opinions and facts and always include the basic characteristics of the security being analyzed in a research report. Expectations based on statistical modeling and analysis are not facts.

Members must explain to clients and prospects the investment decision-making process used.

In preparing recommendations for structured securities, allocation strategies, or any other nontraditional investment, members must communicate those risk factors specific to such investments. In all cases, members should communicate the potential gains and losses on the investment clearly in terms of total returns.

Members must communicate significant changes in the risk characteristics of an investment or investment strategy.

Members must update clients regularly about any changes in the investment process, including any risks and limitations that have been newly identified.

When using projections from quantitative models and analysis, members may violate the Standard by not explaining the limitations of the model and the assumptions it uses, which

provides a context for judging the uncertainty regarding the estimated investment result.

Members and candidates must inform clients about limitations inherent to an investment. Two examples of such limitations are liquidity and capacity. Liquidity refers to the ability to exit an investment readily without experiencing a significant extra cost from doing so. Capacity refers to an investment vehicle's ability to absorb additional investment without reducing the returns it is able to achieve.

## Recommendations for Members

Selection of relevant factors in a report can be a judgment call so members should maintain records indicating the nature of the research, and be able to supply additional information if it is requested by the client or other users of the report.

## Standard V(C) Record Retention

Members and Candidates must develop and maintain appropriate records to support their investment analyses, recommendations, actions, and other investment-related communications with clients and prospective clients.

Members must maintain research records that support the reasons for the analyst's conclusions and any investment actions taken. Such records are the property of the firm. All communications with clients through any medium, including emails and text messages, are records that must be retained.

A member who changes firms must re-create the analysis documentation supporting her recommendation using publicly available information or information obtained from the company and must not rely on memory or materials created at her previous firm.

## Recommendations for Members

If no regulatory standards or firm policies are in place, the Standard recommends a seven-year minimum holding period.

## Recommendations for Firms

This recordkeeping requirement generally is the firm's responsibility.

### MODULE QUIZ 3.6, 3.7

*To best evaluate your performance, enter your quiz answers online.*

1. Connie Fletcher, CFA, works for a small money management firm that specializes in pension accounts. Recently, a friend asked her to act as an unpaid volunteer manager for the city's street sweep pension fund. As part of the position, the city would grant Fletcher a free parking space in front of her downtown office. Before Fletcher accepts, she should *most appropriately*:
   A. do nothing because this is a volunteer position.
   B. inform her current clients in writing and discuss the offer with her employer.
   C. disclose the details of the volunteer position to her employer and obtain written permission from her employer.
2. Sarah Johnson, a portfolio manager, is offered a bonus directly by a client if Johnson meets certain performance goals. To comply with the Standard that governs additional compensation arrangements, Johnson should:
   A. decline to accept a bonus outside of her compensation from her employer.

B. disclose this arrangement to her employer in writing and obtain her employer's permission.
C. disclose this arrangement to her employer only if she actually meets the performance goals and receives the bonus.

3. A member or candidate who has supervisory responsibility:
   A. should place particular emphasis on enforcing investment-related compliance policies.
   B. is responsible for instructing those to whom he has delegated authority about methods to detect and prevent violations of the law and the Code and Standards.
   C. has complied with the Standards if she reports employee violations to upper management and provides a written warning to the employee to cease such activities.
4. Which of the following actions is a *required*, rather than *recommended*, action under the Standard regarding diligence and a reasonable basis for a firm's research recommendations?
   A. Compensate analysts based on a measure of the quality of their research.
   B. Review the assumptions used and evaluate the objectivity of third-party research reports.
   C. Have a policy requiring that research reports and recommendations have a basis that can be substantiated as reasonable and adequate.
5. Claire Marlin, CFA, manages an investment fund specializing in foreign currency trading. Marlin writes a report to investors that describes the basic characteristics of her strategy, which is based on an expected appreciation of the euro relative to other major currencies. Marlin shows the projected returns from the strategy if the euro appreciates less than 5%, between 5% and 10%, or more than 10%, while clearly stating that these forecasts are her opinion. Has Marlin violated the Standard related to communication with clients?
   A. Yes, because she did not include a scenario in which the euro depreciates.
   B. No, because she disclosed the basic characteristics of the investment.
   C. No, because she distinguished fact from opinion and discussed how the strategy may perform under a range of scenarios.
6. If regulations do not specify how long to retain the documents that support an analyst's conclusions, the Code and Standards recommend a period of at least:
   A. 5 years.
   B. 7 years.
   C. 10 years.

# MODULE 3.8: GUIDANCE FOR STANDARD VI

Video covering this content is available online.

## STANDARD VI: CONFLICTS OF INTEREST

### Standard VI(A) Disclosure of Conflicts

Members and Candidates must make full and fair disclosure of all matters that could reasonably be expected to impair their independence and objectivity or interfere with respective duties to their clients, prospective clients, and employer. Members and Candidates must ensure that such disclosures are prominent, are delivered in plain language, and communicate the relevant information effectively.

Members must fully disclose to clients, prospects, and their employers all actual and potential conflicts of interest in order to protect investors and employers. These disclosures must be clearly stated.

The requirement that all potential areas of conflict be disclosed allows clients and prospects to judge motives and potential biases for themselves. Disclosure of broker-dealer market-making activities would be included here. Board service is another area of potential conflict.

The most common conflict that requires disclosure is actual ownership of stock in companies that the member recommends or that clients hold.

Another common source of conflicts of interest is a member's compensation/bonus structure, which can potentially create incentives to take actions that produce immediate gains for the member with little or no concern for longer-term returns for the client. Such conflicts must be disclosed when the member is acting in an advisory capacity and must be updated in the case of significant change in compensation structure.

Members must give their employers enough information to judge the impact of a conflict, take reasonable steps to avoid conflicts, and report them promptly if they occur.

## Recommendations for Members

Any special compensation arrangements, bonus programs, commissions, and incentives should be disclosed.

## Standard VI(B) Priority of Transactions

Investment transactions for clients and employers must have priority over investment transactions in which a Member or Candidate is the beneficial owner.

Client transactions take priority over personal transactions and over transactions made on behalf of the member's firm. Personal transactions include situations where the member is a beneficial owner.

Personal transactions may be undertaken only after clients and the member's employer have had an adequate opportunity to act on a recommendation. Note that family member accounts that are client accounts should be treated just like any client account; they should not be disadvantaged.

Members must not act on information about pending trades for personal gain. The overriding considerations with respect to personal trades are that they do not disadvantage any clients.

When requested, members must fully disclose to investors their firm's personal trading policies.

## Recommendations for Members

Members can avoid conflicts that arise with IPOs by not participating in them.

Members should encourage their firms to adopt the procedures listed in the following recommendations for firms if they have not done so.

## Recommendations for Firms

All firms should have basic procedures in place that address conflicts created by personal investing. The following areas should be included:

- Establish limitations on employee participation in equity IPOs.
- Establish restrictions on participation in private placements. Strict limits should be placed on employee acquisition of these securities and proper supervisory procedures

should be in place. Participation in these investments raises conflict of interest issues similar to those of IPOs.

- Establish blackout/restricted periods. Employees involved in investment decision making should have blackout periods prior to trading for clients—no front running (i.e., purchase or sale of securities in advance of anticipated client or employer purchases and sales). The size of the firm and the type of security should help dictate how severe the blackout requirement should be.
- Establish reporting procedures, including duplicate trade confirmations, disclosure of personal holdings and beneficial ownership positions, and preclearance procedures.

## Standard VI(C) Referral Fees

Members and Candidates must disclose to their employer, clients, and prospective clients, as appropriate, any compensation, consideration, or benefit received from or paid to others for the recommendation of products or services.

Members must inform employers, clients, and prospects of any benefit received for referrals of customers and clients, allowing them to evaluate the full cost of the service as well as any potential partiality. All types of consideration must be disclosed.

### Recommendations for Members

Members should encourage their firms to adopt clear procedures regarding compensation for referrals.

Members should provide their employers with updates at least quarterly.

### Recommendations for Firms

Firms that do not prohibit referral fees should have clear procedures for approval and policies regarding the nature and value of referral compensation received.

# MODULE 3.9: GUIDANCE FOR STANDARD VII

Video covering this content is available online.

# STANDARD VII: RESPONSIBILITIES AS A CFA INSTITUTE MEMBER OR CFA CANDIDATE

## Standard VII(A) Conduct as Participants in CFA Institute Programs

Members and Candidates must not engage in any conduct that compromises the reputation or integrity of CFA Institute or the CFA designation or the integrity, validity, or security of CFA Institute programs.

Members must not engage in any activity that undermines the integrity of the CFA charter. This Standard applies to conduct that includes:

- Cheating on the CFA exam or any exam.
- Revealing anything about either broad or specific topics tested, content of exam questions, or formulas required or not required on the exam.

- Not following rules and policies of the CFA Program.
- Giving confidential information on the CFA Program to candidates or the public.
- Improperly using the designation to further personal and professional goals.
- Misrepresenting information on the Professional Conduct Statement (PCS) or the CFA Institute Professional Development Program.

Members and candidates are not precluded from expressing their opinions regarding the exam program or CFA Institute but must not reveal confidential information about the CFA Program.

Candidates who violate any of the CFA exam policies (e.g., calculator, personal belongings, Candidate Pledge) have violated Standard VII(A).

Members who volunteer in the CFA Program may not solicit or reveal information about questions considered for or included on a CFA exam, about the grading process, or about scoring of questions.

## Standard VII(B) Reference to CFA Institute, the CFA Designation, and the CFA Program

When referring to CFA Institute, CFA Institute membership, the CFA designation, or candidacy in the CFA Program, Members and Candidates must not misrepresent or exaggerate the meaning or implications of membership in CFA Institute, holding the CFA designation, or candidacy in the CFA Program.

Members must not make promotional promises or guarantees tied to the CFA designation, such as over-promising individual competence or over-promising investment results in the future (i.e., higher performance, less risk, etc.).

Members must satisfy these requirements to maintain membership:

- Sign the PCS annually.
- Pay CFA Institute membership dues annually.

If they fail to do this, they are no longer active members.

Do not misrepresent or exaggerate the meaning of the CFA designation.

There is no partial CFA designation. It is acceptable to state that a candidate successfully completed the program in three years if, in fact, he did, but claiming superior ability because of this is not permitted.

## Recommendations for Members

Members should be sure that their firms are aware of the proper references to a member's CFA designation or candidacy, as errors in these references are common.

### MODULE QUIZ 3.8, 3.9

*To best evaluate your performance, enter your quiz answers online.*

1. Daniel Lyons, CFA, is an analyst who covers several stocks including Horizon Company. Lyons's aunt owns 30,000 shares of Horizon. She informs Lyons that she has created a trust in his name into which she has placed 2,000 shares of Horizon. The trust is structured so

that Lyons will not be able to sell the shares until his aunt dies, but may vote the shares. Lyons is due to update his research coverage of Horizon next week. Lyons should *most appropriately*:

A. update the report as usual because he is not a beneficial owner of the stock.
B. advise his superiors that he is no longer able to issue research recommendations on Horizon.
C. disclose the situation to his employer and, if then asked to prepare a report, also disclose his beneficial ownership of the shares in his report.

2. Kate Wilson, CFA, is an equity analyst. Wilson enters two transactions for her personal account. Wilson sells 500 shares of Tibon, Inc., a stock on which her firm currently has a "Buy" recommendation. Wilson buys 200 shares of Hayfield Co. and the following day issues a research report on Hayfield with a "Buy" recommendation. Has Wilson violated the Code and Standards?

A. No.
B. Yes, both of her actions violate the Code and Standards.
C. Yes, but only one of her actions violates the Code and Standards.

3. Hern Investments provides monthly emerging market research to Baker Brokerage in exchange for prospective client referrals and European equity research from Baker. Clients and prospects of Hern are not made aware of the agreement, but clients unanimously rave about the high quality of the research provided by Baker. As a result of the research, many clients with nondiscretionary accounts have earned substantial returns on their portfolios. Managers at Hern have also used the research to earn outstanding returns for the firm's discretionary accounts. Hern has *most likely*:

A. not violated the Code and Standards.
B. violated the Code and Standards by using third-party research in discretionary accounts.
C. violated the Code and Standards by failing to disclose the referral agreement with Baker.

4. After writing the Level I CFA exam, Cynthia White goes to internet discussion site *CFA Haven* to express her frustration. White writes, "CFA Institute is not doing a competent job of evaluating candidates because none of the questions in the June exam touched on Alternative Investments." White *most likely* violated the Standard related to conduct as a candidate in the CFA program by:

A. publicly disputing CFA Institute policies and procedures.
B. disclosing subject matter covered or not covered on a CFA exam.
C. participating in an internet forum that is directed toward CFA Program participants.

5. After passing all three levels of the CFA exams on her first attempts and being awarded her CFA charter, Paula Osgood is promoting her new money management firm by issuing an advertisement. Which of these statements would *most likely* violate the Standard related to use of the CFA designation?

A. "To earn the right to use the CFA designation, Paula passed three exams covering ethics, financial statement analysis, asset valuation, and portfolio management."
B. "Paula passed three 6-hour exams on her first attempts and is a member of her local investment analyst society."
C. "Because of her extensive training, Paula will be able to achieve better investment results than managers who have not been awarded the CFA designation."

# ANSWER KEY FOR MODULE QUIZZES

## Module Quiz 3.1, 3.2, 3.3

1. **C** To comply with Standard I(A) Knowledge of the Law, a member must always abide by the strictest applicable law, regulation, or standard. (Module 3.1, LOS 3.a, 3.b, 3.c)

2. **A** Gifts from clients are acceptable under Standard I(B) Independence and Objectivity, but the Standard requires members and candidates to disclose such gifts to their employers. Standard I(B) allows issuer-paid research as long as the analysis is thorough, independent, unbiased, and has a reasonable and adequate basis for its conclusions, and the compensation from the issuer is disclosed. Members and candidates should consider the potential for conflicts of interest inherent in credit ratings and may need to do independent research to evaluate the soundness of these ratings. (Module 3.1, LOS 3.a, 3.b, 3.c)

3. **C** According to Standard I(C) Misrepresentation, members and candidates must cite the sources of the information they use in their analysis, unless the information is factual data (as opposed to analysis or opinion) from a recognized financial or statistical reporting service. The U.S. Treasury is one example of a recognized source of factual data. (Module 3.2, LOS 3.a, 3.b, 3.c)

4. **B** Failing to act when required by one's professional obligations, such as neglecting to perform due diligence related to an investment recommendation, violates Standard I(D) Misconduct. Acts a member commits outside his professional capacity are misconduct if they reflect poorly on the member or candidate's honesty, integrity, or competence (e.g., theft or fraud).Violations of the law that do not reflect on the member or candidate's honesty, integrity, or competence (e.g., an act related to civil disobedience) are not necessarily regarded as misconduct. (Module 3.2, LOS 3.a, 3.b, 3.c)

5. **A** The statement from the CFO about the current quarter's earnings is material nonpublic information. Ingus violated Standard II(A) Material Nonpublic Information by acting or causing others to act on it. (Module 3.3, LOS 3.a, 3.b, 3.c)

6. **B** The intent of Green Brothers' actions is to manipulate the appearance of market liquidity in order to attract investment to its own funds. The increased trading activity was not based on market fundamentals or an actual trading strategy to benefit investors. It was merely an attempt to mislead market participants in order to increase assets under Green Brothers' management. The action violates Standard II(B) Market Manipulation. (Module 3.3, LOS 3.a, 3.b, 3.c)

## Module Quiz 3.4, 3.5

1. **C** Standard III(A) Loyalty, Prudence, and Care specifies that for the manager of a pension or trust, the duty of loyalty is owed to the beneficiaries, not to the individuals who hired the manager. (Module 3.4, LOS 3.a, 3.b, 3.c)

2. **B** The firm must give its clients an opportunity to act on recommendation changes. Firms can offer different levels of service to clients as long as this is disclosed to all clients. The largest institutional clients would likely be paying higher fees for a greater level of service. The portfolio manager's brother's account should be treated the same as any other client account. (Module 3.4, LOS 3.a, 3.b, 3.c)

3. **A** According to Standard III(C) Suitability, a member or candidate who is in an advisory relationship with a client is responsible for analyzing the suitability of an investment for the client before taking investment action or making a recommendation. If a member or candidate believes an unsolicited trade is unsuitable for a client, the appropriate action is to discuss the trade with the client. The advisor may follow her firm's policies for obtaining client approval if the requested trade would not affect the risk and return of the client's portfolio materially. If the trade would have a material effect, the advisor should discuss with the client whether the IPS needs to be updated. When managing a fund to an index or stated mandate, the manager is responsible for ensuring that potential investments are consistent with the fund's mandate. Suitability for individuals would be a concern for an advisor who recommends the fund to clients, but not for the manager of the fund. (Module 3.5, LOS 3.a, 3.b, 3.c)

4. **C** Recommendations stated in Standard III(D) Performance Presentation include considering the sophistication and knowledge of the audience when presenting performance data. Other recommendations are to include terminated accounts in past performance history; to present the performance of a composite as a weighted average of the performance of similar portfolios, rather than using a single representative account; and to maintain the records and data that were used to calculate performance. (Module 3.5, LOS 3.a, 3.b, 3.c)

5. **B** Members and candidates are required to cooperate with PCP investigations into their own conduct and encouraged to cooperate with PCP investigations into the conduct of others. Sharing confidential information with the PCP is not a violation of Standard III(E) Preservation of Confidentiality. Any client information shared with the PCP will be kept in strict confidence. Standard III(E) states that members and candidates are required to maintain confidentiality of client records even after the end of the client relationship. (Module 3.5, LOS 3.a, 3.b, 3.c)

### Module Quiz 3.6, 3.7

1. **C** According to Standard IV(A) Loyalty, members and candidates are expected to act for the benefit of their employer and not deprive the employer of their skills. Fletcher is performing work similar to the services that her employer provides. Although the position is a volunteer position, Fletcher will receive compensation in the form of a free parking space. In light of the circumstances, Fletcher must disclose the details of the position to her employer and get written permission before accepting the volunteer position. (Module 3.6, LOS 3.a, 3.b, 3.c)

2. **B** Johnson should disclose her additional compensation arrangement in writing to her employer and obtain her employer's written consent before accepting this offer, in accordance with Standard IV(B) Additional Compensation Arrangements. (Module 3.6, LOS 3.a, 3.b, 3.c)

3. **B** Members or candidates may delegate supervisory duties to subordinates but remain responsible for instructing them about how to detect and prevent violations. Reporting the violation and warning the employee are not sufficient to comply with Standard IV(C) Responsibilities of Supervisors. The supervisor must also take steps to prevent further violations while she conducts an investigation, such as limiting the employee's activity or increasing her monitoring of the employee. Supervisors should enforce investment-related and non-investment-related policies equally. (Module 3.6, LOS 3.a, 3.b, 3.c)

4. **B** Standard V(A) Diligence and Reasonable Basis requires analysts who use third-party research to review its assumptions and evaluate the independence and objectivity of the research. The other choices are recommended procedures for compliance with the Standard. (Module 3.7, LOS 3.a, 3.b, 3.c)

5. **A** Standard V(B) Communication with Clients and Prospective Clients requires that members and candidates communicate the risk associated with the investment strategy used and how the strategy is expected to perform in a range of scenarios. These scenarios should include those different from the current trend. Marlin should have discussed how her strategy would perform if the euro depreciates instead of appreciating as she expects. (Module 3.7, LOS 3.a, 3.b, 3.c)

6. **B** When no other regulatory guidance applies, Standard V(C) Record Retention recommends that records be maintained for a minimum of seven years. (Module 3.7, LOS 3.a, 3.b, 3.c)

### Module Quiz 3.8, 3.9

1. **C** Even though the shares are held in trust, Lyons is considered a beneficial owner under Standard VI(A) Disclosure of Conflicts because he has a pecuniary interest in the shares and because has the power to vote the shares. Lyons is obligated to inform his employer of the potential conflict. If Lyons's employer permits him to continue issuing investment recommendations on the company, Lyons must disclose the existence of a potential conflict in his reports. (Module 3.8, LOS 3.a, 3.b, 3.c)

2. **C** Only one of these transactions is a violation. Standard VI(B) Priority of Transactions requires members and candidates to give clients an adequate opportunity to act on a recommendation before trading for accounts in which the member or candidate has a beneficial ownership interest. Members and candidates may trade for their own accounts as long as they do not disadvantage clients, benefit personally from client trades, or violate any regulations that apply. The Standard does not prohibit members and candidates from entering personal transactions that are contrary to what their firms are recommending for clients, as long as the transaction does not violate any of these criteria. (Module 3.8, LOS 3.a, 3.b, 3.c)

3. **C** According to Standard VI(C) Referral Fees, Hern must disclose the referral arrangement between itself and Baker so that potential clients can judge the true cost of Hern's services and assess whether there is any partiality inherent in the recommendation of services. (Module 3.8, LOS 3.a, 3.b, 3.c)

4. **B** Standard VII(A) Conduct as Participants in CFA Institute Programs prohibits candidates from revealing which portions of the Candidate Body of Knowledge were or were not covered on an exam. Members and candidates are free to disagree with the

policies, procedures, or positions taken by the CFA Institute. The Standard does not prohibit participating in CFA Program-related internet blogs, forums, or social networks. (Module 3.9, LOS 3.a, 3.b, 3.c)

5. **C** Standard VII(B) Reference to CFA Institute, the CFA Designation, and the CFA Program prohibits members and candidates from implying superior performance as a result of being a CFA charterholder. Concise factual descriptions of the requirements to obtain the CFA charter are acceptable. Osgood's statement that she passed the exams on her first attempts is acceptable because it states a fact. (Module 3.9, LOS 3.a, 3.b, 3.c)

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2. Ibid.

The following is a review of the Ethical and Professional Standards principles designed to address the learning outcome statements set forth by CFA Institute. Cross-Reference to CFA Institute Assigned Reading #4.

# READING 4: INTRODUCTION TO THE GLOBAL INVESTMENT PERFORMANCE STANDARDS (GIPS®)

**Study Session 1**

## EXAM FOCUS

The following two topic reviews cover the key features of the Global Investment Performance Standards (GIPS®) as adopted by CFA Institute in 1999 and subsequently updated. Compliance with GIPS is voluntary. For the Level I exam, you are responsible for only the "Introduction to the Global Investment Performance Standards (GIPS®)" and the Introduction and Section I (through I.0: Fundamentals of Compliance) of the GIPS document. The GIPS document is included in the book of candidate readings for Level I and is also available on the CFA Institute website. A helpful glossary of terms is included in Section V of the document. Candidates should not underestimate the importance of this material for the exam.

## MODULE 4.1: INTRODUCTION TO GIPS

Video covering this content is available online.

**LOS 4.a: Explain why the GIPS standards were created, what parties the GIPS standards apply to, and who is served by the standards.**

*CFA® Program Curriculum, Volume 1, page 227*

In the past, a variety of reporting procedures were misleading at best. Some of these misleading practices included:

- *Representative accounts*—showing a top-performing portfolio as representative of a firm's results.
- *Survivorship bias*—excluding "weak performance" accounts that have been terminated.
- *Varying time periods*—showing performance for selected time periods with outstanding returns.

GIPS are a set of ethical principles based on a standardized, industry-wide approach. Investment firms can *voluntarily* follow GIPS in their presentation of historical investment results to prospective clients. These standards seek to avoid misrepresentations of performance.

GIPS apply to investment management firms and are intended to serve prospective and existing clients of investment firms. GIPS allow clients to more easily compare investment performance among investment firms and have more confidence in reported performance.

**LOS 4.b: Explain the construction and purpose of composites in performance reporting.**

*CFA® Program Curriculum, Volume 1, page 229*

A composite is a grouping of individual discretionary portfolios representing a similar investment strategy, objective, or mandate. Examples of possible composites are "Large Capitalization Growth Stocks" and "Investment Grade Domestic Bonds." Reporting on the performance of composites gives clients and prospects information about the firm's success in managing various types of securities or results for various investment styles.

A composite, such as International Equities, must include all fee-paying, discretionary portfolios (current and past) that the firm has managed in accordance with this particular strategy. The firm should identify which composite each managed portfolio is to be included in before the portfolio's performance is known. This prevents firms from choosing portfolios to include in a composite in order to create composites with superior returns.

**LOS 4.c: Explain the requirements for verification.**

*CFA® Program Curriculum, Volume 1, page 229*

## *Verification—requirements:*

Verification is performed by a third party, not by the firm itself, on a firm-wide basis. This third party verifier must attest that (1) the firm has complied with all GIPS requirements for composite construction on a firm-wide basis and (2) the firm's processes and procedures are established to present performance in accordance with the calculation methodology required by GIPS, the data requirements of GIPS, and in the format required by GIPS.

## *Verification—recommendations:*

- Firms are encouraged to pursue independent verification. Verification applies to the entire firm's performance measurement practices and methods, not a selected composite.
- Verified firms should include the following disclosure language:

  "[Insert name of firm] has been verified for the periods [insert dates] by [name of verifier]. A copy of the verification report is available upon request."

### MODULE QUIZ 4.1

*To best evaluate your performance, enter your quiz answers online.*

1. Which of the following statements *most accurately* describes the parties that GIPS are intended to apply to and serve? GIPS apply to:
   A. consultants who serve their existing and prospective clients.
   B. firms that issue securities and serve investment management firms.
   C. investment management firms and serve their existing and prospective clients.
2. For a composite to be constructed in compliance with GIPS, the portfolios included in the composite must:
   A. have been managed by the firm for the full performance reporting period.
   B. be selected immediately after the last business day of the period for which the composite's performance will be presented.
   C. include all fee-paying, discretionary portfolios that are managed according to the same strategy, mandate, or investment objective.
3. Verification of compliance with GIPS:
   A. may be performed on single composites.

B. is required for a firm to claim GIPS compliance.
C. requires the verification report to be issued for the entire firm.

## KEY CONCEPTS

### LOS 4.a

GIPS are principles investment firms can voluntarily follow. They are designed to avoid misrepresentations of historical investment results to clients and prospects. GIPS allow clients to more easily compare investment performance among investment firms and have more confidence in reported performance.

### LOS 4.b

A composite is a grouping of discretionary portfolios representing a similar investment strategy, objective, or mandate. A composite must include all portfolios (current and past) that the firm has managed in accordance with this particular strategy. The firm should identify which composite each managed portfolio is to be included in before the portfolio's performance is known.

### LOS 4.c

Firms are encouraged to pursue independent verification of GIPS compliance. If they seek verification, it must be performed by a third party, not by the firm itself.

The third-party verifier must attest that (1) the firm has complied with all GIPS requirements for composite construction on a firm-wide basis, and (2) the firm's processes and procedures are established to present performance in accordance with the calculation methodology required by GIPS, the data requirements of GIPS, and in the format required by GIPS.

# ANSWER KEY FOR MODULE QUIZ

## Module Quiz 4.1

1. **C** GIPS apply to investment management firms. They are intended to serve prospective and existing clients of investment firms and consultants who advise these clients. (LOS 4.a)

2. **C** To comply with GIPS, a composite must include all fee-paying, discretionary portfolios managed according to the same investment objective, strategy, or mandate. The composite or composites in which a portfolio will be included must be determined on an ex-ante basis (i.e., before the period for which the composite's performance will be calculated). A GIPS-compliant composite must include terminated accounts. (LOS 4.b)

3. **C** Verification of GIPS compliance is optional, but if a firm chooses to seek third-party verification, the report must be issued with respect to the whole firm. GIPS verification cannot be carried out for a single composite. (LOS 4.c)

The following is a review of the Ethical and Professional Standards principles designed to address the learning outcome statements set forth by CFA Institute. Cross-Reference to CFA Institute Assigned Reading #5.

# READING 5: GLOBAL INVESTMENT PERFORMANCE STANDARDS (GIPS®)

**Study Session 1**

## MODULE 5.1: THE GIPS STANDARDS

Video covering this content is available online.

**LOS 5.a: Describe the key features of the GIPS standards and the fundamentals of compliance.**

*CFA® Program Curriculum, Volume 1, page 235*

## GIPS Objectives

- To obtain global acceptance of calculation and presentation standards in a fair, comparable format with full disclosure.
- To ensure consistent, accurate investment performance data in areas of reporting, records, marketing, and presentations.
- To promote fair competition among investment management firms in all markets without unnecessary entry barriers for new firms.
- To promote global "self-regulation."

## Key Characteristics of GIPS

- To claim compliance, an investment management firm must define its "firm." This definition should reflect the "distinct business entity" that is held out to clients and prospects as the investment firm.
- GIPS are ethical standards for performance presentation which ensure fair representation of results and full disclosure.
- Include all actual fee-paying, discretionary portfolios in composites for a minimum of five years or since firm or composite inception. After presenting five years of compliant data, the firm must add annual performance each year going forward up to a minimum of 10 years.
- Firms are required to use certain calculation and presentation standards and make specific disclosures.
- Input data must be accurate.
- GIPS contain both *required* and *recommended* provisions—firms are *encouraged* to adopt the *recommended* provisions.
- Firms are encouraged to present all pertinent additional and supplemental information.
- There will be no partial compliance and only full compliance can be claimed.

- Follow the local laws for cases in which a local or country-specific law or regulation conflicts with GIPS, but disclose the conflict.
- Certain recommendations may become requirements in the future.
- Supplemental private equity and real estate provisions contained in GIPS are to be applied to those asset classes.

**Fundamentals of compliance** contain both requirements and recommendations.

## *Definition of the firm—requirements:*

- Apply GIPS on a firm-wide basis.
- Firm must be defined as a distinct business unit.
- Total firm assets include total market value of discretionary and nondiscretionary assets, including fee-paying and non-fee-paying accounts.
- Include asset performance of sub-advisors, as long as the firm has discretion over sub-advisor selection.
- If a firm changes its organization, historical composite results cannot be changed.

## *Definition of the firm—recommendations:*

- Include the broadest definition of the firm, including all geographical offices marketed under the same brand name.

## *Document policies and procedures—requirements:*

- Document, in writing, policies and procedures the firm uses to comply with GIPS.

## *Claim of compliance—requirements:*

- Once GIPS requirements have been met, the following compliance statement must be used:
  "[Insert name of firm] has prepared and presented this report in compliance with the Global Investment Performance Standards (GIPS®)."
- There is no such thing as partial compliance.
- There are to be no statements referring to calculation methodologies used in a composite presentation as being "in accordance with GIPS" or the like.
- Similarly, there should be no such statements referring to the performance of an individual, existing client as being "calculated in accordance with GIPS" or the like, unless a compliant firm is reporting results directly to the client.

## *Firm fundamental responsibilities—requirements:*

- Firms must provide a compliant presentation to *all* prospects (prospect must have received a presentation within the previous 12 months).
- Provide a composite list and composite description to all prospects that make a request. List discontinued composites for at least five years.

- Provide, to clients requesting it, a compliant presentation and a composite description for any composite included on the firm's list.
- When jointly marketing with other firms, if one of the firms claims GIPS compliance, be sure it is clearly defined as separate from noncompliant firms.
- Firms are encouraged to comply with recommendations and must comply with all requirements. Be aware of updates, guidance statements, and the like.

### LOS 5.b: Describe the scope of the GIPS standards with respect to an investment firm's definition and historical performance record.

*CFA® Program Curriculum, Volume 1, page 236*

The definition of the firm, for purposes of GIPS compliance, must be the corporation, subsidiary, or division that is held out to clients as a business entity. If a firm has different geographic locations (e.g., all doing business under the name of Bluestone Advisers), then the definition of the firm should include all the various geographic locations and their clients. Firms based in any country may present GIPS-compliant performance histories.

A firm must initially present a minimum of five years of compliant performance presentation for the firm and each composite unless the firm or composite has been in existence less than five years. For firms or composites in existence less than five years, compliant performance since inception must be presented in order to claim compliance. After the initial compliant performance presentation, one year of compliant performance must be added each year to a required (minimum) performance history of 10 years.

Firms may present periods of noncompliant performance immediately prior to the compliant performance history as long as no noncompliant performance is presented for any periods after January 1, 2000. Firms must specify which performance results are noncompliant and the ways in which such (noncompliant) performance does not comply with GIPS.

### LOS 5.c: Explain how the GIPS standards are implemented in countries with existing standards for performance reporting and describe the appropriate response when the GIPS standards and local regulations conflict.

*CFA® Program Curriculum, Volume 1, page 237*

Firms that previously presented performance in compliance with a particular Country Version of GIPS (CVG) may claim GIPS compliance for any CVG-compliant results prior to January 1, 2006. Firms that report such CVG-compliant performance data must continue to include that performance data in subsequent GIPS-compliant presentations until a minimum of 10 years of compliant performance is presented.

In any cases where country-specific regulations conflict with GIPS, firms must follow the applicable country-specific regulations but must also disclose the nature of the conflict with GIPS.

### LOS 5.d: Describe the nine major sections of the GIPS standards.

*CFA® Program Curriculum, Volume 1, page 239*

0. *Fundamentals of Compliance*. The fundamental issues involved in complying with GIPS are (a) definition of the firm, (b) documentation of firm policies and procedures with respect

to GIPS compliance, (c) complying with GIPS updates, (d) claiming compliance in the appropriate manner, and (e) appropriate verification statement when a third-party verifier is employed.

To claim GIPS compliance, a firm *must*:

- Comply with all GIPS requirements, guidance, and updates.
- Comply with all applicable laws and regulations concerning performance presentation.
- Apply GIPS on a firmwide basis.
- Document their policies for maintaining GIPS compliance.
- Make reasonable efforts to provide all prospective clients a GIPS-compliant presentation.
- Provide a complete list of composites and their descriptions to any prospective client who requests one. The list must include all current composites and any that were terminated within the last five years.
- Provide a GIPS-compliant performance presentation for any composite to any prospective client who requests one.
- Define the firm as the distinct business entity that is held out to clients or prospective clients.
- Calculate total firm assets as the fair value of all assets managed by the firm, including discretionary, nondiscretionary, fee-paying, and non-fee-paying accounts.
- Clearly define the firm that is claiming GIPS compliance, when marketing jointly with other firms.

In addition, a GIPS-compliant firm *must not*:

- Present false or misleading performance information.
- Claim any kind of partial GIPS compliance.
- State that a specific calculation is in compliance with GIPS.
- State that an individual portfolio's performance has been calculated in compliance with GIPS, except when reporting a single client's portfolio performance to that client.
- Choose which prospective clients will receive a GIPS-compliant presentation.
- Alter historical performance of composites based on a change in the firm's organization.

1. *Input Data*. Input data should be consistent in order to establish full, fair, and comparable investment performance presentations.

2. *Calculation Methodology*. Certain methodologies are required for portfolio return calculations and certain other methodologies are required for composite return calculations. Uniformity in methods across firms is required so that their results are comparable.

3. *Composite Construction*. Creation of meaningful, asset-weighted composites is important to achieve a fair presentation. Composite performance is based on the performance of one or more portfolios that have the same investment strategy or investment objective. Composite returns are the asset-weighted average (not a simple average) of the returns on the portfolios that are included in each composite.

4. *Disclosures*. The firm must disclose information about the presentation and the policies adopted by the firm so that the raw numbers presented in the report are understandable to the user. There are some disclosures that all firms must make, but some disclosures may not apply to all firms. If a disclosure is not applicable to a specific firm, the firm is not required to include any statement regarding it.

5. *Presentation and Reporting*. Investment performance must be presented according to GIPS requirements. Other firm-specific information not specifically required by GIPS should be included when appropriate.

6. *Real Estate*. Certain provisions apply to all real estate investments (land, buildings, etc.) regardless of the level of control the firm has over management of the investment. These provisions apply regardless of whether the asset is producing revenue or there is leverage involved in the investment.

7. *Private Equity*. Private equity investments must be valued according to the GIPS Private Equity Valuation Principles, which are contained in Appendix D, unless the investment is an open-end or evergreen fund (which must follow regular GIPS). Private equity investments include all investment in companies that are not publicly traded, regardless of their stage of business development. This would include venture capital investments, ownership of a previously public company that has been purchased (taken private), and mezzanine financing, as well as limited partnership shares in such investments and fund-of-funds investments.

8. *Wrap Fee/Separately Managed Account (SMA) Portfolios*. For these portfolios, some of the requirements and recommendations in Sections 0 through 5 are supplemented or replaced by the requirements specified in this section.

## MODULE QUIZ 5.1

*To best evaluate your performance, enter your quiz answers online.*

1. Firms that claim to present investment performance in compliance with GIPS are required to:
   A. document the procedures and policies they use to ensure GIPS compliance.
   B. clearly disclose any calculations or other aspects of a presentation that are noncompliant with GIPS.
   C. calculate total firm assets as the fair value of all fee-paying discretionary portfolios managed by the firm.
2. Which of the following definitions of a firm would violate GIPS?
   A. Investment firm that has been in existence for less than five years.
   B. Regional branch of an investment management firm marketed under the name of its parent.
   C. Entity registered with the national regulator that oversees its investment management activities.
3. If a country has regulations in place that conflict with GIPS, firms that wish to claim GIPS compliance:
   A. may not do so because GIPS do not permit exceptions or partial compliance.
   B. must establish a subsidiary in a location where local law does not conflict with GIPS.
   C. must comply with local regulations and disclose the nature of the conflict in the presentation.
4. Which of the following includes only sections of the Global Investment Performance Standards?
   A. Disclosure, Public Equity, Presentation and Reporting.
   B. Real Estate, Calculation Methodology, Fundamentals of Compliance.
   C. Input Data, Composite Construction, Wrap Fee/Speculative Margin Account (SMA) Portfolios.

## KEY CONCEPTS

### LOS 5.a

GIPS contain both required and recommended provisions. Certain recommendations may become requirements in the future.

Firms must include all fee-paying, discretionary portfolios in composites for a minimum of five years or since firm or composite inception.

Total firm assets include the total market value of discretionary and nondiscretionary assets, including fee-paying and non-fee-paying accounts.

There is no such thing as partial compliance. There are to be no statements referring to calculation methodologies used in a composite presentation as being "in accordance with GIPS." Similarly, there should be no statements referring to the performance of an individual client as being "calculated in accordance with GIPS" unless a compliant firm is reporting results directly to the client.

### LOS 5.b

GIPS compliance must be firmwide, where the "firm" is the distinct business entity that is held out to clients and prospects.

A firm's initial GIPS-compliant historical performance record must include a minimum of five years of historical data, or since firm or composite inception. After the five years has been achieved, the firm must add one additional year of performance each year, up to a minimum of 10 years.

Non-GIPS-compliant information may be linked to compliant history, but no noncompliant performance can be presented after January 1, 2000. Firms must clearly identify the noncompliant portion of results.

### LOS 5.c

Firms that presented performance in compliance with a Country Version of GIPS (CVG) may claim GIPS compliance for any CVG-compliant results prior to January 1, 2006.

If country-specific regulations conflict with GIPS, firms must follow the applicable country-specific regulations but must also disclose the nature of the conflict with GIPS.

### LOS 5.d

The nine sections of GIPS are:

0. Fundamentals of Compliance. These are issues for firms to consider when claiming GIPS compliance (e.g., firm definition).
1. Input Data. Input data should be consistent in order to establish full, fair, and comparable investment performance presentations.
2. Calculation Methodology. Certain uniform methodologies are required for portfolio and composite return calculations.
3. Composite Construction. Creation of meaningful, asset-weighted composites is important to achieve a fair presentation.

4. Disclosures. Certain information must be disclosed about the presentation and the policies adopted by the firm.
5. Presentation and Reporting. Investment performance must be presented according to GIPS requirements, and when appropriate, other firm-specific information should be included.
6. Real Estate. These provisions apply to all real estate investments regardless of the level of control the firm has.
7. Private Equity. These must be valued according to the GIPS Private Equity Valuation Principles, unless it is an open-end or evergreen fund (which must follow regular GIPS).
8. Wrap Fee/Separately Managed Account (SMA) Portfolios. Certain special GIPS standards apply to these.

# ANSWER KEY FOR MODULE QUIZ

## Module Quiz 5.1

1. **A** GIPS Section 0, "Fundamentals of Compliance," states that firms must document their procedures for remaining in compliance with GIPS. Firms which state that they are in compliance with GIPS must comply fully and may not claim partial compliance. GIPS defines total firm assets as the fair value of all portfolios managed by the firm, including discretionary and nondiscretionary, fee-paying and non-fee-paying portfolios. (LOS 5.a)

2. **B** The definition of a firm for GIPS-compliant performance presentation should include all geographical offices marketed under the same name brand. (LOS 5.b)

3. **C** Firms must always comply with the laws and regulations of the country in which they reside. In cases where local regulations conflict with GIPS, a firm can still claim GIPS compliance if they disclose the conflict fully in an otherwise compliant presentation. (LOS 5.c)

4. **B** The nine major sections of GIPS are (0) Fundamentals of Compliance; (1) Input Data; (2) Calculation Methodology; (3) Composite Construction; (4) Disclosure; (5) Presentation and Reporting; (6) Real Estate; (7) Private Equity; and (8) Wrap Fee/Separately Managed Account (SMA) Portfolios. The incorrect choices misstate at least one of these sections. Neither "Public Equity" nor "Speculative Margin Account" is a section of GIPS. (LOS 5.d)

# TOPIC ASSESSMENT: ETHICAL AND PROFESSIONAL STANDARDS

*You have now finished the Ethical and Professional Standards topic section. The following Topic Assessment provides immediate feedback on how effective your study has been for this material. The number of questions on this test is equal to the number of questions for the topic on one-half of the actual Level I CFA exam. Questions are more exam-like than typical Module Quiz or QBank questions; a score of less than 70% indicates that your study likely needs improvement. These tests are best taken timed; allow 1.5 minutes per question.*

*After you've completed this Topic Assessment, you may additionally log in to your Schweser.com online account and enter your answers in the Topic Assessments product. Select "Performance Tracker" to view a breakdown of your score. Select "Compare with Others" to display how your score on the Topic Assessment compares to the scores of others who entered their answers.*

1. Karen Jones, CFA, is an outside director for Valley Manufacturing. At a directors' meeting, Jones finds out that Valley has made several contributions to foreign politicians that she suspects were illegal. Jones checks with her firm's legal counsel and determines that the contributions were indeed illegal. At the next board meeting, Jones urges the board to disclose the contributions. The board, however, votes not to make a disclosure. Jones's *most appropriate* action would be to:
   A. protest the board's actions in writing to the executive officer of Valley.
   B. resign from the board and seek legal counsel as to her legal disclosure requirements.
   C. inform her supervisor of her discovery and cease attending meetings until the matter is resolved.

2. Over the past two days, Lorraine Quigley, CFA, who is the manager of a hedge fund, has been purchasing large quantities of Craeger Industrial Products' common stock while at the same time shorting put options on the same stock. Quigley did not notify her clients of the trades, although they are aware of the fund's general strategy to generate returns. Which of the following statements is *most likely* correct? Quigley:
   A. did not violate the Code and Standards.
   B. violated the Code and Standards by manipulating the prices of publicly traded securities.
   C. violated the Code and Standards by failing to disclose the transactions to clients before they occurred.

3. Melvin Byrne, CFA, manages a portfolio for James Martin, a wealthy client. Martin's portfolio is well diversified with a slight tilt toward capital appreciation. Martin requires very little income from the portfolio. Recently, Martin's brother, Cliff, has become a client of Byrne. Byrne proceeds to invest Cliff's portfolio in a similar manner to James's portfolio based on the fact that both brothers have a similar lifestyle and are only two years apart in age. Which of the following statements is *most likely* correct? Byrne:
   A. violated the Code and Standards by knowingly creating a conflict of interest between James's and Cliff's portfolios.

B. violated the Code and Standards by failing to determine Cliff's objectives and constraints prior to investing his portfolio.
C. did not violate the Code and Standards.

4. Robert Blair, CFA, is a director of research and has had an ongoing battle with his firm's management about the adequacy of its compliance system. Blair believes the firm's compliance procedures are inadequate in that they are not being monitored or carefully followed. Blair should *most appropriately*:
   A. resign from the firm unless the compliance system is strengthened and followed.
   B. send his superior a memo outlining the problem.
   C. decline in writing to continue to accept supervisory responsibility until reasonable compliance procedures are adopted.

5. Beth Anderson, CFA, is a portfolio manager for several wealthy clients, including Reuben Carlyle. Anderson manages Carlyle's personal portfolio of stock and bond investments. Carlyle recently told Anderson that he is under investigation for tax evasion related to his business, Carlyle Concrete. After learning about the investigation, Anderson informs a friend at a local investment bank so that the bank may withdraw their proposal to take Carlyle Concrete public. Anderson has *most likely*:
   A. violated the Code and Standards by failing to maintain the confidentiality of her client's information.
   B. violated the Code and Standards by failing to detect and report the tax evasion to the proper authorities.
   C. not violated the Code and Standards since the information she conveyed pertained to illegal activities on the part of her client.

6. A research report states: "Based on the fact that the South American utilities sector has seen rapid growth in new service orders, we expect that most companies in the sector will be able to convert the revenue increases into significant profits. We also believe the trend will continue for the next three to five years." The report goes on to describe the major risks of investing in this market. The author of this report:
   A. has not violated the Code and Standards.
   B. violated the Code and Standards by failing to properly distinguish factual information from opinions.
   C. violated the Code and Standards by failing to properly identify details related to the operations of South American utilities.

7. Frist Investments, Inc., has just hired Michael Pulin to manage institutional portfolios, most of which are pension related. Pulin has just taken the Level III CFA exam and is awaiting his results. Pulin has more than 15 years of investment management experience with individual clients but has never managed an institutional portfolio. Pulin joined the CFA Institute as an affiliate member two years ago and is in good standing with the organization. Which of the following statements would be *most appropriate* for Frist to use in advertising Pulin as a new member of the firm? Pulin:
   A. has many years of investment experience, which, along with his participation in the CFA program, will allow him to deliver superior investment performance relative to other managers.
   B. is a CFA Level III and passed the first two exams on the first attempt. He is an affiliate member of the CFA Institute. We expect him to become a regular member if he passes the Level III examination.

C. is a Level III CFA candidate and has many years of excellent performance in the investment management industry. Pulin is an affiliate member of the CFA Institute and will be eligible to become a CFA charterholder and regular member if he passes the Level III CFA Exam.

8. Zanuatu, an island nation, does not have any regulations precluding the use of nonpublic information. Alfredo Romero has a friend and fellow CFA charterholder there, Donna Gordon, with whom he has shared nonpublic information regarding firms outside his industry. The information concerns several firms' internal earnings and cash flow projections. Gordon may:
   A. trade on the information under the laws of Zanuatu, which govern her behavior.
   B. not trade on the information under CFA Institute Standards, which govern her behavior.
   C. trade on the information under CFA Institute Standards, since the firms concerned are outside Romero's industry.

9. Samantha Donovan, CFA, is an exam proctor for the Level II CFA exam. The day before the exam is to be administered, Donovan faxes a copy of one of the questions to two friends, James Smythe and Lynn Yeats, who are Level II candidates in the CFA program. Donovan, Smythe, and Yeats had planned the distribution of an exam question months in advance. Smythe used the fax to prepare for the exam. Yeats, however, had second thoughts and threw the fax away without looking at its contents. Which of the following statements is *most likely* correct?
   A. Smythe violated the Code and Standards, but Yeats did not.
   B. Donovan violated the Code and Standards, but Smythe did not.
   C. Donovan and Yeats both violated the Code and Standards.

10. Sally Albright, CFA, works full time for Frank & Company, an investment management firm, as a fixed-income security analyst. Albright has been asked by a business contact at KDG Enterprises to accept some analytical work from KDG on a consulting basis. The work would entail investigating potential distressed debt securities in the small-cap market. Albright should *most appropriately*:
    A. accept the work as long as she obtains consent to all the terms of the engagement from Frank & Company.
    B. not accept the work as it violates the Code and Standards by creating a conflict of interest.
    C. accept the work as long as she obtains written consent from KDG and does it on her own time.

11. Josef Karloff, CFA, acts as liaison between Pinnacle Financial (an investment management firm) and Summit Inc. (an investment banking boutique specializing in penny stocks). When Summit underwrites an IPO, Karloff routinely has Pinnacle issue vague statements implying that the firm has cash flows, financial resources, and growth prospects that are better than is the case in reality. This action is *most likely* a violation of the section of the Standards concerning:
    A. fair dealing.
    B. nonpublic information.
    C. misconduct.

12. Steve Matthews, CFA, is a principal at Carlson Brothers, a leading regional investment bank specializing in initial public offerings of small to mid-sized biotech firms. Just

before many of the IPOs are offered to the general public, Matthews arranges for 10% of the shares to be distributed to management of the firm going public. This action is a violation of the Standard concerning:

A. additional compensation.
B. disclosure of conflicts of interest.
C. fair dealing.

13. Will Hunter, CFA, is a portfolio manager at NV Asset Managers. An investment banker asks Hunter to purchase shares in a new IPO to support the price long enough for insiders to liquidate their holdings. Hunter realizes that the price of the shares will almost certainly fall dramatically after his buying support ceases. NV management strongly suggests that Hunter honor the investment banker's request since NV has had a longstanding relationship with the investment bank. If Hunter agrees to make the purchases, he will:
    A. not violate the Code and Standards.
    B. violate the Standard concerning market manipulation.
    C. violate the Standard concerning priority of transactions.

14. Neiman Investment Co. receives brokerage business from Pick Asset Management in exchange for referring prospective clients to Pick. Pick advises clients—in writing, before the relationship is established—of the nature of its arrangement with Neiman. With regard to this practice, Pick has:
    A. complied with the Code and Standards.
    B. violated the Code and Standards by failing to preserve the confidentiality of the agreement with Neiman.
    C. violated the Code and Standards by participating in an agreement that creates a conflict of interest.

15. Ralph Salley, a Level I candidate in the CFA Program, is explaining Standard VI(B) Priority of Transactions to his supervisor. Salley states, "The Standard recommends, but does not require, that members and candidates should not participate in initial public offerings. The Standard also recommends that trades for accounts of family members be made after those for other clients, but before those for the account of the members and candidates responsible for executing the transactions." Salley's explanation of the Standard is:
    A. correct.
    B. incorrect, because the Standard does not recommend that trades for family members be made after those for other clients.
    C. incorrect, because the Standard requires that members and candidates not participate in initial public offerings.

16. Which of the following policies is required to comply with GIPS?
    A. Compliance with GIPS must be verified by an independent third party.
    B. Restructuring of the firm's organization cannot be used as a basis to change the historical performance results of a composite.
    C. The definition of the firm must include the firm's offices in all countries and regions.

17. Mason Smith is trying to decide which of the following composite definitions, submitted by his junior analysts, would be considered an acceptable composite according to GIPS. An acceptable composite can include all:

A. accounts that are managed directly from the firm's Hong Kong office.
B. actively managed portfolios, but exclude passively managed portfolios.
C. portfolios that are managed to provide a return approximately equal to that of the S&P 500 Index.

18. An investment management firm, Investco, Inc., was recently audited by the United States Securities and Exchange Commission (SEC). Investco included the following statement in its performance presentation report: "This report has been verified as GIPS compliant by Investco's Compliance Department and the United States Securities and Exchange Commission." Does this constitute acceptable verification under GIPS?
    A. No, because only one party may perform GIPS verification.
    B. No, because neither party involved in the audit constitutes an acceptable GIPS verification.
    C. Yes, because an audit was performed implicitly by the SEC and explicitly by the firm's audit team.

# TOPIC ASSESSMENT ANSWERS: ETHICAL AND PROFESSIONAL STANDARDS

*Questions in this Topic Assessment reference LOS 2.b, c and LOS 3.a, b, c, except where noted.*

1. **B** According to Standard I(A) Knowledge of the Law, since she has taken steps to stop the illegal activities and the board has ignored her, Jones must dissociate from the board and seek legal advice as to what other actions would be appropriate in this instance. She may need to inform legal or regulatory authorities of the illegal activities. (Study Session 1, Modules 2.1, 3.1)

2. **A** Quigley's trades are most likely an attempt to take advantage of an arbitrage opportunity that exists between Craeger's common stock and its put options. She is not manipulating the prices of securities in an attempt to mislead market participants, which would violate Standard II(B) Market Manipulation. She is pursuing a legitimate investment strategy. Participants in her hedge fund are aware of the fund's investment strategy, and thus Quigley did not violate the Code and Standards by not disclosing this specific set of trades in advance of trading. (Study Session 1, Modules 2.1, 3.3)

3. **B** Standard III(C) Suitability requires that before taking investment action, members and candidates must make a reasonable inquiry into a client's or prospect's investment objectives and constraints as well as their prior investment experience. Byrne cannot assume that because the brothers have similar lifestyles and are close in age that they should have similarly managed portfolios. Byrne should have interviewed Cliff directly before investing his portfolio. (Study Session 1, Modules 2.1, 3.5)

4. **C** According to Standard IV(C) Responsibilities of Supervisors, because he is aware that the firm's compliance procedures are not being monitored and followed and because he has repeatedly tried to get company management to correct the situation, Blair should decline supervisory responsibility until adequate procedures to detect and prevent violations of laws, regulations, and the Code and Standards are adopted and followed. If he does not do so, he will be in violation of the Code and Standards. (Study Session 1, Modules 2.1, 3.6)

5. **A** Anderson must maintain the confidentiality of client information according to Standard III(E) Preservation of Confidentiality. Confidentiality may be broken in instances involving illegal activities on the part of the client, but the client's information may only be relayed to proper authorities. Anderson did not have the right to inform the investment bank of her client's investigation. (Study Session 1, Modules 2.1, 3.5)

6. **A** Historical growth can be cited as a fact. The report states that the firm expects further growth and profitability, which phrases the statement as an opinion. Under Standard V(B) Communication with Clients and Prospective Clients, it is not necessary to include every detail about a potential investment in a report. Members and candidates are expected to use their judgment and identify the most important factors to include. (Study Session 1, Modules 2.1, 3.7)

7. **C** Standard VII(B) governs acceptable methods of referencing the CFA Institute, CFA designation, and CFA Program. Candidates may reference their candidacy if they are enrolled for, or waiting for the results of, a CFA exam. Pulin may also reference his membership status with the CFA Institute as well as his remaining eligibility requirements to become a CFA charterholder. (Study Session 1, Modules 2.1, 3.9)

8. **B** Even though the laws of Zanuatu would not preclude trading on the information, as a CFA charterholder, Gordon is bound by the CFA Institute Code and Standards. Standard II(A) prohibits the use of material nonpublic information, and Gordon may not trade the stocks about which she has such information under any circumstances. (Study Session 1, Modules 2.1, 3.1, 3.3)

9. **C** In this situation, Donovan, Smythe, and Yeats all violated Standard VII(A) Conduct as Participants in CFA Institute Programs. The Standard prohibits conduct that compromises the integrity, validity, or security of the CFA exams. Donovan clearly breached the exam security. Smythe and Yeats both compromised the integrity of the exams by planning to use the actual exam question to gain an advantage over other candidates. Even though Yeats did not ultimately use the information to study for the exam, she participated in a scheme to cheat on the CFA exam. (Study Session 1, Modules 2.1, 3.9)

10. **A** Albright may accept work for which she receives outside compensation and that may compete with her employer only if she obtains her employer's consent. Under Standard IV(A) Loyalty, she must obtain her employer's consent for her consulting work before beginning the work. (Study Session 1, Modules 2.1, 3.6)

11. **C** Since the statements are vague, we have no direct evidence that a violation of securities law has occurred. However, under Standard I(D) Misconduct, members and candidates are prohibited from engaging in activities involving deceit. Karloff's action is a clear attempt to mislead the investing public regarding the value of Summit IPOs. (Study Session 1, Modules 2.1, 3.2)

12. **C** Standard III(B) Fair Dealing requires that members not use their position to disadvantage clients, specifically in the case of IPOs. (Study Session 1, Modules 2.1, 3.4)

13. **B** NV management is asking Hunter to violate Standard II(B) Market Manipulation, which prohibits actions that are designed to distort prices or artificially increase trading volume. The intent is to mislead market participants and allow corporate insiders to take advantage of artificially high prices. (Study Session 1, Modules 2.1, 3.3)

14. **A** Pick has not violated the Standards. The referral arrangement is fully disclosed to clients before they agree to do business with Pick, in accordance with Standard VI(C) Referral Fees. Therefore, clients can fully assess the effect of the agreement on the referral and how the agreement may affect their accounts before hiring Pick as their asset manager. (Study Session 1, Modules 2.1, 3.8)

15. **B** Standard VI(B) Priority of Transactions recommends that members and candidates avoid the appearance of conflict of interest by not participating in IPOs. If a family member is a fee-paying client, the member or candidate should treat them like any other client, not giving any advantage or disadvantage to their accounts. (Study Session 1, Modules 2.1, 3.8)

16. **B** Firms cannot alter historical performance records of composites simply because of a reorganization of the firm. This is required by GIPS Standard 0.A.15. Third-party verification of GIPS compliance and including all geographical offices in the definition of the firm are recommended, but not required. (Study Session 1, Modules 4.1, 5.1, LOS 4.c, 5.a, 5.b)

17. **C** Composites are groups of portfolios that represent a similar investment strategy, objective, or mandate. Clearly, grouping all portfolios managed to mirror the S&P 500 Index return constitutes a composite according to this definition. Organizing composites by office or by a generic active management category is not acceptable, as these categories do not reflect any sort of strategy, objective, or mandate. (Study Session 1, Module 4.1, LOS 4.b)

18. **B** GIPS verification must be performed by an independent third party. The SEC audit does not constitute a GIPS verification. (Study Session 1, Module 4.1, LOS 4.c)

The following is a review of the Quantitative Methods (1) principles designed to address the learning outcome statements set forth by CFA Institute. Cross-Reference to CFA Institute Assigned Reading #6.

# READING 6: THE TIME VALUE OF MONEY

**Study Session 2**

## EXAM FOCUS

This topic review covers time value of money concepts and applications. Procedures are presented for calculating the future value and present value of a single cash flow, an annuity, and a series of uneven cash flows. The impact of different compounding periods is examined, along with the procedures for solving for other variables in time value of money problems. Your main objective in this chapter is to master time value of money mechanics (i.e., learn how to crunch the numbers). Work all the questions and problems found at the end of this review. Make sure you know how to grind out all the time value of money problems on your calculator. The more rapidly you can do them (correctly), the more time you will have for the more conceptual parts of the exam.

## MODULE 6.1: EAY AND COMPOUNDING FREQUENCY

Video covering this content is available online.

The concept of **compound interest** or **interest on interest** is deeply embedded in time value of money (TVM) procedures. When an investment is subjected to compound interest, the growth in the value of the investment from period to period reflects not only the interest earned on the original principal amount but also on the interest earned on the previous period's interest earnings—the interest on interest.

TVM applications frequently call for determining the **future value** (FV) of an investment's cash flows as a result of the effects of compound interest. Computing FV involves projecting the cash flows forward, on the basis of an appropriate compound interest rate, to the end of the investment's life. The computation of the **present value** (PV) works in the opposite direction—it brings the cash flows from an investment back to the beginning of the investment's life based on an appropriate compound rate of return.

Being able to measure the PV and/or FV of an investment's cash flows becomes useful when comparing investment alternatives because the value of the investment's cash flows must be measured at some common point in time, typically at the end of the investment horizon (FV) or at the beginning of the investment horizon (PV).

### Using a Financial Calculator

It is very important that you be able to use a financial calculator when working TVM problems because the exam is constructed under the assumption that candidates have the ability to do so. There is simply no other way that you will have time to solve TVM problems. *CFA Institute allows only two types of calculators to be used for the exam—the TI BAII Plus® (including the BAII Plus Professional) and the HP 12C® (including the HP 12C Platinum).* This topic review is written primarily with the TI BAII Plus in mind. If you don't

already own a calculator, go out and buy a TI BAII Plus! However, if you already own the HP 12C and are comfortable with it, by all means continue to use it.

The TI BAII Plus comes preloaded from the factory with the periods per year function (P/Y) set to 12. This automatically converts the annual interest rate (I/Y) into monthly rates. While appropriate for many loan-type problems, this feature is not suitable for the vast majority of the TVM applications we will be studying. So prior to using our SchweserNotes™, please set your P/Y key to "1" using the following sequence of keystrokes:

[2nd] [P/Y] "1" [ENTER] [2nd] [QUIT]

As long as you do not change the P/Y setting, it will remain set at one period per year until the battery from your calculator is removed (it does not change when you turn the calculator on and off). If you want to check this setting at any time, press [2nd] [P/Y]. The display should read P/Y = 1.0. If it does, press [2nd] [QUIT] to get out of the "programming" mode. If it doesn't, repeat the procedure previously described to set the P/Y key. With P/Y set to equal 1, it is now possible to think of I/Y as the interest rate per compounding period and N as the number of compounding periods under analysis. Thinking of these keys in this way should help you keep things straight as we work through TVM problems.

Before we begin working with financial calculators, you should familiarize yourself with your TI by locating the TVM keys noted below. These are the only keys you need to know to work virtually all TVM problems.

- N = Number of compounding periods.
- I/Y = Interest rate per compounding period.
- PV = Present value.
- FV = Future value.
- PMT = Annuity payments, or constant periodic cash flow.
- CPT = Compute.

**PROFESSOR'S NOTE**

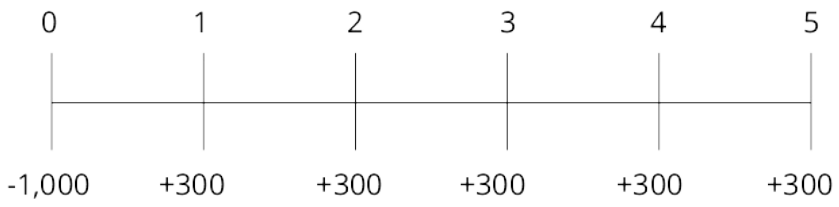
We have provided an online video in the Resource Library on how to use the TI calculator. You can view it by logging in at www.Schweser.com.

## Time Lines

It is often a good idea to draw a time line before you start to solve a TVM problem. A **time line** is simply a diagram of the cash flows associated with a TVM problem. A cash flow that occurs in the present (today) is put at time zero. Cash outflows (payments) are given a negative sign, and cash inflows (receipts) are given a positive sign. Once the cash flows are assigned to a time line, they may be moved to the beginning of the investment period to calculate the PV through a process called **discounting** or to the end of the period to calculate the FV using a process called **compounding**.

Figure 6.1 illustrates a time line for an investment that costs $1,000 today (outflow) and will return a stream of cash payments (inflows) of $300 per year at the end of each of the next five years.

**Figure 6.1: Time Line**




Please recognize that the cash flows occur at the end of the period depicted on the time line. Furthermore, note that the end of one period is the same as the beginning of the next period. For example, the end of the second year (t = 2) is the same as the beginning of the third year, so a cash flow at the beginning of Year 3 appears at time t = 2 on the time line. Keeping this convention in mind will help you keep things straight when you are setting up TVM problems.

**PROFESSOR'S NOTE**

Throughout the problems in this review, rounding differences may occur between the use of different calculators or techniques presented in this document. So don't panic if you are a few cents off in your calculations.

### LOS 6.a: Interpret interest rates as required rates of return, discount rates, or opportunity costs.

*CFA® Program Curriculum, Volume 1, page 314*

Interest rates are our measure of the time value of money, although risk differences in financial securities lead to differences in their equilibrium interest rates. Equilibrium interest rates are the **required rate of return** for a particular investment, in the sense that the market rate of return is the return that investors and savers require to get them to willingly lend their funds. Interest rates are also referred to as **discount rates** and, in fact, the terms are often used interchangeably. If an individual can borrow funds at an interest rate of 10%, then that individual should *discount* payments to be made in the future at that rate in order to get their equivalent value in current dollars or other currency. Finally, we can also view interest rates as the **opportunity cost** of current consumption. If the market rate of interest on 1-year securities is 5%, earning an additional 5% is the opportunity forgone when current consumption is chosen rather than saving (postponing consumption).

### LOS 6.b: Explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk.

*CFA® Program Curriculum, Volume 1, page 315*

The **real risk-free rate** of interest is a theoretical rate on a single-period loan that has no expectation of inflation in it. When we speak of a real rate of return, we are referring to an investor's increase in purchasing power (after adjusting for inflation). Since expected inflation in future periods is not zero, the rates we observe on U.S. Treasury bills (T-bills), for example, are risk-free rates but not *real* rates of return. T-bill rates are *nominal risk-free rates* because they contain an *inflation premium*. The approximate relation here is:

nominal risk-free rate = real risk-free rate + expected inflation rate

Securities may have one or more **types of risk**, and each added risk increases the required rate of return on the security. These types of risk are:

- **Default risk**. The risk that a borrower will not make the promised payments in a timely manner.
- **Liquidity risk**. The risk of receiving less than fair value for an investment if it must be sold for cash quickly.
- **Maturity risk**. As we will cover in detail in the section on debt securities, the prices of longer-term bonds are more volatile than those of shorter-term bonds. Longer maturity bonds have more maturity risk than shorter-term bonds and require a maturity risk premium.

Each of these risk factors is associated with a risk premium that we add to the nominal risk-free rate to adjust for greater default risk, less liquidity, and longer maturity relative to a very liquid, short-term, default risk-free rate such as that on T-bills. We can write:

required interest rate on a security = nominal risk-free rate
+ default risk premium
+ liquidity premium
+ maturity risk premium

### LOS 6.c: Calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding.

*CFA® Program Curriculum, Volume 1, page 321*

Financial institutions usually quote rates as stated annual interest rates, along with a compounding frequency, as opposed to quoting rates as periodic rates—the rate of interest earned over a single compounding period. For example, a bank will quote a savings rate as 8%, compounded quarterly, rather than 2% per quarter. The rate of interest that investors actually realize as a result of compounding is known as the **effective annual rate** (EAR) or **effective annual yield** (EAY). EAR represents the annual rate of return actually being earned *after adjustments have been made for different compounding periods*.

EAR may be determined as follows:

$$\text{EAR} = (1 + \text{periodic rate})^{m} - 1$$

where:
periodic rate = stated annual rate/m
m = the number of compounding periods per year

Obviously, the EAR for a stated rate of 8% *compounded annually* is not the same as the EAR for 8% *compounded semiannually*, or *quarterly*. Indeed, whenever compound interest is being used, the stated rate and the actual (effective) rate of interest are equal only when interest is compounded annually. Otherwise, the greater the compounding frequency, the greater the EAR will be in comparison to the stated rate.

The computation of EAR is necessary when comparing investments that have different compounding periods. It allows for an apples-to-apples rate comparison.

**EXAMPLE: Computing EAR**

Compute EAR if the stated annual rate is 12%, compounded quarterly.

**Answer:**

Here m = 4, so the periodic rate is $\frac{12}{4} = 3\%$.

Thus, EAR = $(1 + 0.03)^4 - 1 = 1.1255 - 1 = 0.1255 = 12.55\%$.

This solution uses the [$y^x$] key on your financial calculator. The exact keystrokes on the TI for the above computation are 1.03 [$y^x$] 4 [=]. On the HP, the strokes are 1.03 [ENTER] 4 [$y^x$].

**EXAMPLE: Computing EARs for a range of compounding frequencies**

Using a stated rate of 6%, compute EARs for semiannual, quarterly, monthly, and daily compounding.

**Answer:**

EAR with:

| | | |
|---|---|---|
| semiannual compounding | $= (1 + 0.03)^2 - 1$ | $= 1.06090 - 1 = 0.06090 = 6.090\%$ |
| quarterly compounding | $= (1 + 0.015)^4 - 1$ | $= 1.06136 - 1 = 0.06136 = 6.136\%$ |
| monthly compounding | $= (1 + 0.005)^{12} - 1$ | $= 1.06168 - 1 = 0.06168 = 6.168\%$ |
| daily compounding | $= (1 + 0.00016438)^{365} - 1$ | $= 1.06183 - 1 = 0.06183 = 6.183\%$ |

Notice here that the EAR increases as the compounding frequency increases.

**PROFESSOR'S NOTE**

The limit of shorter and shorter compounding periods is called continuous compounding, which we will address in a later topic review.

## LOS 6.d: Solve time value of money problems for different frequencies of compounding.

*CFA® Program Curriculum, Volume 1, page 321*

When compounding periods are other than annual, we must take account of that fact in our calculations. Consider the following two examples.

**EXAMPLE: Growth with quarterly compounding**

John plans to invest $2,500 in an account that will earn 8% per year with quarterly compounding. How much will be in the account at the end of two years?

**Answer:**

There are eight quarterly compounding periods in two years, and the effective quarterly rate is 8 / 4 = 2%. The account will grow to $2{,}500(1.02)^8 = \$2{,}929.15$. Alternatively, since the EAR is $1.02^4 - 1 = 0.082432$, we can grow the $2,500 at 8.2432% for two years to get $2{,}500(1.082432)^2 = \$2{,}929.15$, which is the same result.

**EXAMPLE: Present value with monthly compounding**

Alice would like to have $5,000 saved in an account at the end of three years. If the return on the account is 9% per year with monthly compounding, how much must Alice deposit today in order to reach her savings goal in three years?

**Answer:**

The effective monthly rate is 9 / 12 = 0.75%, and we can calculate the present value of $5,000 three years (36 months) from now as $5{,}000 / (1.0075)^{36} = \$3{,}820.74$. Alternatively, since the EAR is $1.0075^{12} - 1 =$

0.093807, we can calculate the present value by discounting 5,000 at the EAR for three years. 5,000 / $1.093807^3$ = $3,820.74, which is the same result.

### MODULE QUIZ 6.1

*To best evaluate your performance, enter your quiz answers online.*

1. An interest rate is *best* interpreted as:
   A. a discount rate or a measure of risk.
   B. a measure of risk or a required rate of return.
   C. a required rate of return or the opportunity cost of consumption.
2. An interest rate from which the inflation premium has been subtracted is known as:
   A. a real interest rate.
   B. a risk-free interest rate.
   C. a real risk-free interest rate.
3. What is the effective annual rate for a credit card that charges 18% compounded monthly?
   A. 15.38%.
   B. 18.81%.
   C. 19.56%.
4. Given daily compounding, the growth of $5,000 invested for one year at 12% interest will be *closest* to:
   A. $5,600.
   B. $5,628.
   C. $5,637.

## MODULE 6.2: CALCULATING PV AND FV

Video covering this content is available online.

**LOS 6.e: Calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows.**

*CFA® Program Curriculum, Volume 1, page 316*

## Future Value of a Single Sum

*Future value* is the amount to which a current deposit will grow over time when it is placed in an account paying compound interest. The FV, also called the compound value, is simply an example of compound interest at work.

The formula for the FV of a *single* cash flow is:

$$FV = PV(1 + I/Y)^N$$

where:
PV = amount of money invested today (the present value)
I/Y = rate of return per compounding period
N = total number of compounding periods

In this expression, the investment involves a single cash outflow, PV, which occurs today, at t = 0 on the time line. The single sum FV formula will determine the value of an investment at the end of N compounding periods, given that it can earn a fully compounded rate of return, I/Y, over all of the periods.

The factor $(1 + I/Y)^N$ represents the compounding rate on an investment and is frequently referred to as the **future value factor**, or the **future value interest factor**, for a single cash flow at I/Y over N compounding periods. These are the values that appear in interest factor tables, which we will not be using.

**EXAMPLE: FV of a single sum**

Calculate the FV of a $200 investment at the end of two years if it earns an annually compounded rate of return of 10%.

**Answer:**

To solve this problem with your calculator, input the relevant data and compute FV.

N = 2; I/Y = 10; PV = –200; CPT → FV = $242

**PROFESSOR'S NOTE**

Note the negative sign on PV. This is not necessary, but it makes the FV come out as a positive number. If you enter PV as a positive number, ignore the negative sign that appears on the FV.

This relatively simple problem could also be solved using the following equation:

$$FV = 200(1 + 0.10)^2 = \$242$$

On the TI calculator, enter 1.10 [$y^x$] 2 [×] 200 [=].

## Present Value of a Single Sum

The PV of a single sum is today's value of a cash flow that is to be received at some point in the future. In other words, it is the amount of money that must be invested today, at a given rate of return over a given period of time, in order to end up with a specified FV. As previously mentioned, the process for finding the PV of a cash flow is known as *discounting* (i.e., future cash flows are "discounted" back to the present). The interest rate used in the discounting process is commonly referred to as the **discount rate** but may also be referred to as the **opportunity cost**, **required rate of return**, and the **cost of capital**. Whatever you want to call it, it represents the annual compound rate of return that can be earned on an investment.

The relationship between PV and FV can be seen by examining the FV expression stated earlier. Rewriting the FV equation in terms of PV, we get:

$$PV = FV \times \left[\frac{1}{(1+I/Y)^N}\right] = \frac{FV}{(1+I/Y)^N}$$

Note that for a single future cash flow, PV is always less than the FV whenever the discount rate is positive.

The quantity $1 / (1 + I/Y)^N$ in the PV equation is frequently referred to as the **present value factor**, **present value interest factor**, or **discount factor** for a single cash flow at I/Y over N compounding periods.

**EXAMPLE: PV of a single sum**

Given a discount rate of 10%, calculate the PV of a $200 cash flow that will be received in two years.

**Answer:**

To solve this problem, input the relevant data and compute PV.

N = 2; I/Y = 10; FV = 200; CPT → PV = –$165.29 (ignore the sign)

### PROFESSOR'S NOTE

With single sum PV problems, you can either enter FV as a positive number and ignore the negative sign on PV or enter FV as a negative number.

This relatively simple problem could also be solved using the following PV equation:

$$\text{PV} = \frac{200}{(1+0.10)^2} = \$165.29$$

On the TI, enter 1.10 [$y^x$] 2 [=] [1/x] [×] 200 [=].

The PV computed here implies that at a rate of 10%, an investor will be indifferent between $200 in two years and $165.29 today. Put another way, $165.29 is the amount that must be invested today at a 10% rate of return in order to generate a cash flow of $200 at the end of two years.

## Annuities

An **annuity** is a stream of *equal cash flows* that occurs at *equal intervals* over a given period. Receiving $1,000 per year at the end of each of the next eight years is an example of an annuity. There are two types of annuities: **ordinary annuities** and **annuities due**. The *ordinary annuity* is the most common type of annuity. It is characterized by cash flows that occur at the *end* of each compounding period. This is a typical cash flow pattern for many investment and business finance applications. The other type of annuity is called an *annuity due*, where payments or receipts occur at the beginning of each period (i.e., the first payment is today at t = 0).

Computing the FV or PV of an annuity with your calculator is no more difficult than it is for a single cash flow. You will know four of the five relevant variables and solve for the fifth (either PV or FV). The difference between single sum and annuity TVM problems is that instead of solving for the PV or FV of a single cash flow, we solve for the PV or FV of a stream of equal periodic cash flows, where the size of the periodic cash flow is defined by the payment (PMT) variable on your calculator.

**EXAMPLE: FV of an ordinary annuity**

What is the future value of an ordinary annuity that pays $200 per year at the end of each of the next three years, given the investment is expected to earn a 10% rate of return?

**Answer:**

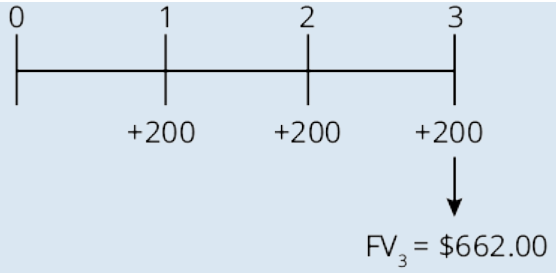
This problem can be solved by entering the relevant data and computing FV.

N = 3; I/Y = 10; PMT = –200; CPT → FV = $662

Implicit here is that PV = 0; clearing the TVM functions sets both PV and FV to zero.

The time line for the cash flows in this problem is depicted in the following figure.

**FV of an Ordinary Annuity**




As indicated here, the sum of the compounded values of the individual cash flows in this three-year ordinary annuity is $662. Note that the annuity payments themselves amounted to $600, and the balance is the interest earned at the rate of 10% per year.

To find the PV of an ordinary annuity, we use the future cash flow stream, PMT, that we used with FV annuity problems, but we discount the cash flows back to the present (time = 0) rather than compounding them forward to the terminal date of the annuity.

Here again, the PMT variable is a *single* periodic payment, *not* the total of all the payments (or deposits) in the annuity. The $PVA_O$ measures the collective PV of a stream of equal cash flows received at the end of each compounding period over a stated number of periods, N, given a specified rate of return, I/Y. The following examples illustrate how to determine the PV of an ordinary annuity using a financial calculator.

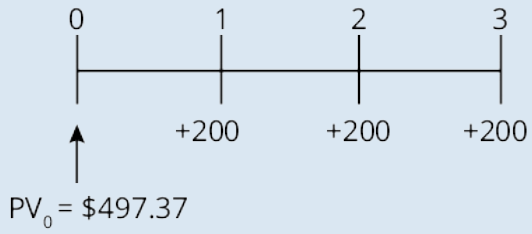
**EXAMPLE: PV of an ordinary annuity**

What is the PV of an annuity that pays $200 per year at the end of each of the next three years, given a 10% discount rate?

**Answer:**

The payments occur at the end of the year, so this annuity is an ordinary annuity. To solve this problem, enter the relevant information and compute PV.

N = 3; I/Y = 10; PMT = –200; FV = 0; CPT → PV = $497.37




The $497.37 computed here represents the amount of money that an investor would need to invest *today* at a 10% rate of return to generate three end-of-year cash flows of $200 each.

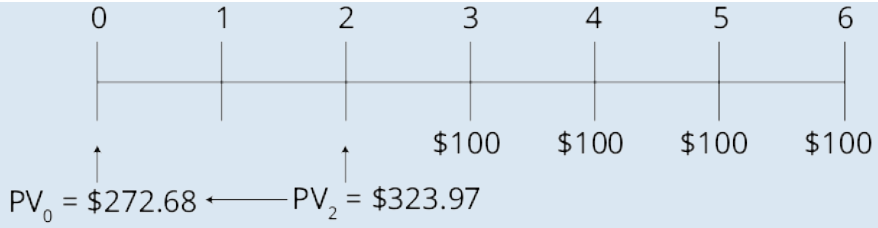
**EXAMPLE: PV of an ordinary annuity beginning later than t = 1**

What is the present value of four $100 end-of-year payments if the first payment is to be received three years from today and the appropriate rate of return is 9%?

**Answer:**

The time line for this cash flow stream is shown in the following figure.

**PV of an Annuity Beginning at t = 3**




*Step 1:* Find the present value of the annuity as of the end of year 2 ($PV_2$).

Input the relevant data and solve for $PV_2$.

N = 4; I/Y = 9; PMT = –100; FV = 0; CPT → PV = $PV_2$ = $323.97

*Step 2:* Find the present value of $PV_2$.

Input the relevant data and solve for $PV_0$.

N = 2; I/Y = 9; PMT = 0; FV = –323.97; CPT → PV = $PV_0$ = $272.68

In this solution, the annuity was treated as an ordinary annuity. The PV was computed one period before the first payment, and we discounted $PV_2$ = $323.97 over two years. We need to stress this important point. The PV annuity function on your calculator set in "END" mode gives you the value *one period before the annuity begins*. Although the annuity begins at t = 3, we discounted the result for only two periods to get the present (t = 0) value.

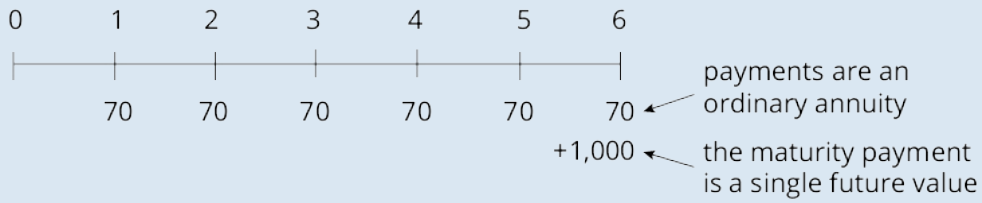
**EXAMPLE: PV of a bond's cash flows**

A bond will make coupon interest payments of 70 euros (7% of its face value) at the end of each year and will also pay its face value of 1,000 euros at maturity in six years. If the appropriate discount rate is 8%, what is the present value of the bond's promised cash flows?

**Answer:**

The six annual coupon payments of 70 euros each can be viewed as an ordinary annuity. The maturity value of 1,000 euros is the future value of the bond at the time the last coupon payment is made. On a time line, the promised payment stream is as shown below.

**Cash flows for a 6-year, 7%, 1,000 euro bond**




The PV of the bond's cash flows can be broken down into the PV of a 6-payment ordinary annuity, plus the PV of a 1,000 euro lump sum to be received six years from now.

The calculator solution is:

N = 6; PMT = 70; I/Y = 8; FV = 1,000; CPT PV = –953.77

With a yield to maturity of 8%, the value of the bond is 953.77 euros.

Note that the PMT and FV must have the same sign, since both are cash flows paid to the investor (paid by the bond issuer). The calculated PV will have the opposite sign from PMT and FV.

## Future Value of an Annuity Due

Sometimes it is necessary to find the *FV of an annuity due* ($FVA_D$), an annuity where the annuity payments (or deposits) occur at the beginning of each compounding period.

Fortunately, our financial calculators can be used to do this, but with one slight modification —the calculator must be set to the beginning-of-period (BGN) mode. To switch between the BGN and END modes on the TI, press [2nd] [BGN] [2nd] [SET]. When this is done, "BGN" will appear in the upper right corner of the display window. If the display indicates the desired mode, press [2nd] [QUIT]. You will normally want your calculator to be in the ordinary annuity (END) mode, so remember to switch out of BGN mode after working annuity due problems. Note that nothing appears in the upper right corner of the display window when the TI is set to the END mode. It should be mentioned that while annuity due payments are made or received at the beginning of each period, the FV of an annuity due is calculated as of the end of the last period.

Another way to compute the FV of an annuity due is to calculate the FV of an ordinary annuity, and simply multiply the resulting FV by [1 + periodic compounding rate (I/Y)]. Symbolically, this can be expressed as:

$$FVA_D = FVA_O \times (1 + I/Y)$$

The following examples illustrate how to compute the FV of an annuity due.

**EXAMPLE: FV of an annuity due**

What is the future value of an annuity that pays $200 per year at the beginning of each of the next three years, commencing today, if the cash flows can be invested at an annual rate of 10%?

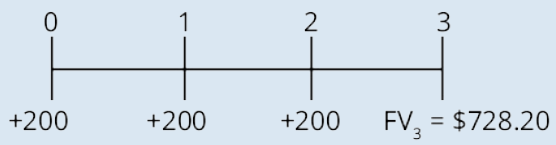
**Answer:**

Note in the time line in the following figure that the FV is computed as of the end of the last year in the life of the annuity, Year 3, even though the final payment occurs at the beginning of Year 3 (end of Year 2).

To solve this problem, put your calculator in the BGN mode ([2nd] [BGN] [2nd] [SET] [2nd] [QUIT] on the TI or [g] [BEG] on the HP), then input the relevant data and compute FV.

N = 3; I/Y = 10; PMT = –200; CPT → FV = $728.20

**FV of an Annuity Due**




Alternatively, we could calculate the FV for an ordinary annuity and multiply it by (1 + I/Y). Leaving your calculator in the END mode, enter the following inputs:

N = 3; I/Y = 10; PMT = –200; CPT → $FVA_O$ = $662.00

$FVA_D = FVA_O \times (1 + I/Y) = 662 \times 1.10 = \$728.20$

## Present Value of an Annuity Due

While less common than those for ordinary annuities, some problems may require you to find the *PV of an annuity due* ($PVA_D$). Using a financial calculator, this really shouldn't be much of a problem. With an annuity due, *there is one less discounting period* since the first cash flow occurs at t = 0 and thus is already its PV. This implies that, all else equal, the PV of an annuity due will be greater than the PV of an ordinary annuity.

As you will see in the next example, there are two ways to compute the PV of an annuity due. The first is to put the calculator in the BGN mode and then input all the relevant variables (PMT, I/Y, and N) as you normally would. The second, and far easier way, is to treat the cash flow stream as an ordinary annuity over N compounding periods, and simply multiply the resulting PV by [1 + periodic compounding rate (I/Y)].

Symbolically, this can be stated as:

$$PVA_D = PVA_O \times (1 + I/Y)$$

The advantage of this second method is that you leave your calculator in the END mode and won't run the risk of forgetting to reset it. Regardless of the procedure used, the computed PV is given as of the beginning of the first period, t = 0.

**EXAMPLE: PV of an annuity due**

Given a discount rate of 10%, what is the present value of an annuity that makes $200 payments at the beginning of each of the next three years, starting today?

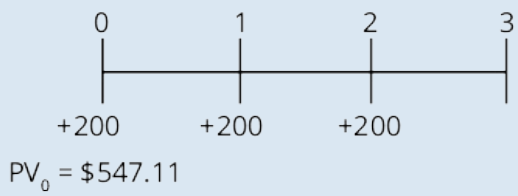
**Answer:**

First, let's solve this problem using the calculator's BGN mode. Set your calculator to the BGN mode ([2nd] [BGN] [2nd] [SET] [2nd] [QUIT] on the TI or [g] [BEG] on the HP), enter the relevant data, and compute PV.

$$N = 3;\ I/Y = 10;\ PMT = -200;\ CPT \rightarrow PVA_D = \$547.11$$

The time line for this problem is shown in the following figure.

**PV of an Annuity Due**




Alternatively, this problem can be solved by leaving your calculator in the END mode. First, compute the PV of an ordinary 3-year annuity. Then multiply this PV by (1 + I/Y). To use this approach, enter the relevant inputs and compute PV.

$$N = 3;\ I/Y = 10;\ PMT = -200;\ CPT \rightarrow PVA_O = \$497.37$$

$$PVA_D = PVA_O \times (1 + I/Y) = \$497.37 \times 1.10 = \$547.11$$

## Present Value of a Perpetuity

A **perpetuity** is a financial instrument that pays a fixed amount of money at set intervals over an *infinite* period of time. In essence, a perpetuity is a perpetual annuity. Most preferred stocks are examples of perpetuities since they promise fixed interest or dividend payments forever. Without going into all the excruciating mathematical details, the discount factor for a perpetuity is just one divided by the appropriate rate of return (i.e., 1/r). Given this, we can compute the PV of a perpetuity.

$$PV_{\text{perpetuity}} = \frac{PMT}{I/Y}$$

The PV of a perpetuity is the fixed periodic cash flow divided by the appropriate periodic rate of return.

As with other TVM applications, it is possible to solve for unknown variables in the $PV_{perpetuity}$ equation. In fact, you can solve for any one of the three relevant variables, given the values for the other two.

**EXAMPLE: PV of a perpetuity**

Kodon Corporation issues preferred stock that will pay $4.50 per year in annual dividends beginning next year and plans to follow this dividend policy forever. Given an 8% rate of return, what is the value of Kodon's preferred stock today?

**Answer:**

Given that the value of the stock is the PV of all future dividends, we have:

$$PV_{perpetuity} = \frac{4.50}{0.08} = \$56.25$$

Thus, if an investor requires an 8% rate of return, the investor should be willing to pay $56.25 for each share of Kodon's preferred stock. Note that the PV of a perpetuity is its value one period before its next payment.

**EXAMPLE: PV of a deferred perpetuity**

Assume the Kodon preferred stock in the preceding examples is scheduled to pay its first dividend in four years, and is non-cumulative (i.e., does not pay any dividends for the first three years). Given an 8% required rate of return, what is the value of Kodon's preferred stock today?

**Answer:**

As in the previous example, $PV_{perpetuity} = \frac{4.50}{0.08} = \$56.25$, but because the first dividend is paid at t = 4, this PV is the value at t = 3. To get the value of the preferred stock today, we must discount this value for three periods: $\frac{56.25}{(1.08)^3} = \$44.65$.

## MODULE QUIZ 6.2

*To best evaluate your performance, enter your quiz answers online.*

1. The amount an investor will have in 15 years if $1,000 is invested today at an annual interest rate of 9% will be *closest* to:
   A. $1,350.
   B. $3,518.
   C. $3,642.
2. How much must be invested today, at 8% interest, to accumulate enough to retire a $10,000 debt due seven years from today?
   A. $5,835.
   B. $6,123.
   C. $8,794.
3. An investor has just won the lottery and will receive $50,000 per year at the end of each of the next 20 years. At a 10% interest rate, the present value of the winnings is *closest* to:
   A. $425,678.
   B. $637,241.
   C. $2,863,750.
4. An investor is to receive a 15-year, $8,000 annuity, with the first payment to be received today. At an 11% discount rate, this annuity's worth today is *closest* to:
   A. $55,855.

B. $57,527.
C. $63,855.

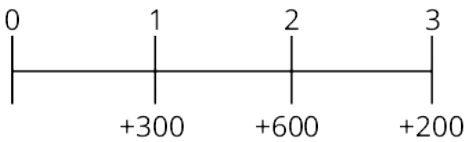
5. If $1,000 is invested today and $1,000 is invested at the beginning of each of the next three years at 12% interest (compounded annually), the amount an investor will have at the end of the fourth year will be *closest* to:
   A. $4,779.
   B. $5,353.
   C. $6,792.
6. Terry Corporation preferred stocks are expected to pay a $9 annual dividend forever. If the required rate of return on equivalent investments is 11%, a share of Terry preferred should be worth:
   A. $81.82.
   B. $99.00.
   C. $122.22.

## MODULE 6.3: UNEVEN CASH FLOWS

Video covering this content is available online.

It is not uncommon to have applications in investments and corporate finance where it is necessary to evaluate a cash flow stream that is not equal from period to period. The time line in Figure 6.2 depicts such a cash flow stream.

**Figure 6.2: Time Line for Uneven Cash Flows**




This three-year cash flow series is not an annuity since the cash flows are different every year. In essence, this series of uneven cash flows is nothing more than a stream of annual single sum cash flows. Thus, to find the PV or FV of this cash flow stream, all we need to do is sum the PVs or FVs of the individual cash flows.

**EXAMPLE: Computing the FV of an uneven cash flow series**

Using a rate of return of 10%, compute the future value of the three-year uneven cash flow stream described above at the end of the third year.

**Answer:**

The FV for the cash flow stream is determined by first computing the FV of each individual cash flow, then summing the FVs of the individual cash flows.

$FV_1$: PV = –300; I/Y = 10; N = 2; CPT → FV = $FV_1$ = 363

$FV_2$: PV = –600; I/Y = 10; N = 1; CPT → FV = $FV_2$ = 660

$FV_3$: PV = –200; I/Y = 10; N = 0; CPT → FV = $FV_3$ = 200

**FV of cash flow stream = $\Sigma FV_{individual}$ = 1,223**

**EXAMPLE: Computing PV of an uneven cash flow series**

Compute the present value of this three-year uneven cash flow stream described previously using a 10% rate of return.

**Answer:**

This problem is solved by first computing the PV of each individual cash flow, then summing the PVs of the individual cash flows, which yields the PV of the cash flow stream. Again the signs of the cash flows are preserved.

$PV_1$: FV = 300; I/Y = 10; N = 1; CPT → PV = $PV_1$ = –272.73

$PV_2$: FV = 600; I/Y = 10; N = 2; CPT → PV = $PV_2$ = –495.87

$PV_3$: FV = 200; I/Y = 10; N = 3; CPT → PV = $PV_3$ = –150.26

**PV of cash flow stream = $\Sigma PV_{individual}$ = $918.86**

## Solving Time Value of Money Problems When Compounding Periods Are Other Than Annual

While the conceptual foundations of TVM calculations are not affected by the compounding period, more frequent compounding does have an impact on FV and PV computations. Specifically, since an increase in the frequency of compounding increases the effective rate of interest, it also *increases* the FV of a given cash flow and *decreases* the PV of a given cash flow.

**EXAMPLE: The effect of compounding frequency on FV and PV**

Compute the FV one year from now of $1,000 today and the PV of $1,000 to be received one year from now using a stated annual interest rate of 6% with a range of compounding periods.

**Answer:**

**Compounding Frequency Effect**

| Compounding Frequency | Interest Rate per Period | Effective Annual Rate | Future Value | Present Value |
|---|---|---|---|---|
| Annual (m = 1) | 6.000% | 6.00% | $1,060.00 | $943.396 |
| Semiannual (m = 2) | 3.000 | 6.090 | 1,060.90 | 942.596 |
| Quarterly (m = 4) | 1.500 | 6.136 | 1,061.36 | 942.184 |
| Monthly (m = 12) | 0.500 | 6.168 | 1,061.68 | 941.905 |
| Daily (m = 365) | 0.016438 | 6.183 | 1,061.83 | 941.769 |

There are two ways to use your financial calculator to compute PVs and FVs under different compounding frequencies:

1. Adjust the number of periods per year (P/Y) mode on your calculator to correspond to the compounding frequency (e.g., for quarterly, P/Y = 4). WE DO NOT RECOMMEND THIS APPROACH!
2. Keep the calculator in the annual compounding mode (P/Y = 1) and enter I/Y as the interest rate per compounding period, and N as the number of compounding periods in the investment horizon. Letting *m* equal the number of compounding periods per year, the basic formulas for the calculator input data are determined as follows:

   I/Y = the annual interest rate / m

   N = the number of years × m

The computations for the FV and PV amounts in the previous example are:

FV = –1,000; I/Y = 6 / 1 = 6; N = 1 × 1 = 1:

$PV_A$: CPT → PV = $PV_A$ = 943.396

$PV_S$: FV = –1,000; I/Y = 6 / 2 = 3; N = 1 × 2 = 2:
CPT → PV = $PV_S$ = 942.596

$PV_Q$: FV = –1,000; I/Y = 6 / 4 = 1.5; N = 1 × 4 = 4:
CPT → PV = $PV_Q$ = 942.184

$PV_M$: FV = –1,000; I/Y = 6 / 12 = 0.5; N = 1 × 12 = 12:
CPT → PV = $PV_M$ = 941.905

$PV_D$: FV = –1,000; I/Y = 6 / 365 = 0.016438; N = 1 × 365 = 365:
CPT → PV = $PV_D$ = 941.769

$FV_A$: PV = –1,000; I/Y = 6 / 1 = 6; N = 1 × 1 = 1:
CPT → FV = $FV_A$ = 1,060.00

$FV_S$: PV = –1,000; I/Y = 6 / 2 = 3; N = 1 × 2 = 2:
CPT → FV = $FV_S$ = 1,060.90

$FV_Q$: PV = –1,000; I/Y = 6 / 4 = 1.5; N = 1 × 4 = 4:
CPT → FV = $FV_Q$ =1,061.36

$FV_M$: PV = –1,000; I/Y = 6 / 12 = 0.5; N = 1 × 12 = 12:
CPT → FV = $FV_M$ = 1,061.68

$FV_D$: PV = –1,000; I/Y = 6 / 365 = 0.016438; N = 1 × 365 = 365:
CPT → FV = $FV_D$ = 1,061.83

**EXAMPLE: FV of a single sum using quarterly compounding**

Compute the FV of $2,000 today, five years from today using an interest rate of 12%, compounded quarterly.

**Answer:**

To solve this problem, enter the relevant data and compute FV:

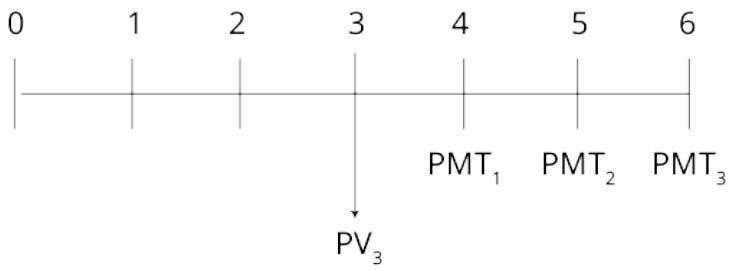
N = 5 × 4 = 20; I/Y = 12 / 4 = 3; PV = –$2,000; CPT → FV = $3,612.22

## LOS 6.f: Demonstrate the use of a time line in modeling and solving time value of money problems.

*CFA® Program Curriculum, Volume 1, page 316*

In most of the PV problems we have discussed, cash flows were discounted back to the current period. In this case, the PV is said to be indexed to t = 0, or the time index is t = 0. For example, the PV of a 3-year ordinary annuity that is indexed to t = 0 is computed at the beginning of Year 1 (t = 0). Contrast this situation with another 3-year ordinary annuity that doesn't start until Year 4 and extends to Year 6. It would not be uncommon to want to know the PV of this annuity at the beginning of Year 4, in which case the time index is t = 3. The time line for this annuity is presented in Figure 6.3.

**Figure 6.3: Indexing Time Line to Other Than t = 0**




The following examples will illustrate how to compute I/Y, N, or PMT in annuity problems.

**EXAMPLE: Computing an annuity payment needed to achieve a given FV**

At an expected rate of return of 7%, how much must be deposited at the end of each year for the next 15 years to accumulate $3,000?

**Answer:**

To solve this problem, enter the three relevant known values and compute PMT.

N = 15; I/Y = 7; FV = +$3,000; CPT → PMT = –$119.38 (ignore sign)

**EXAMPLE: Computing a loan payment**

Suppose you are considering applying for a $2,000 loan that will be repaid with equal end-of-year payments over the next 13 years. If the annual interest rate for the loan is 6%, how much will your payments be?

**Answer:**

The size of the end-of-year loan payment can be determined by inputting values for the three known variables and computing PMT.

N = 13; I/Y = 6; PV = –2,000; CPT → PMT = $225.92

**EXAMPLE: Computing the number of periods in an annuity**

How many $100 end-of-year payments are required to accumulate $920 if the discount rate is 9%?

**Answer:**

The number of payments necessary can be determined by inputting the relevant data and computing N.

I/Y = 9%; FV = $920; PMT = –$100; CPT → N = 7 years

It will take seven annual $100 payments, compounded at 9% annually, to accrue an investment value of $920.

## PROFESSOR'S NOTE

Remember the sign convention. PMT and FV must have opposite signs or your calculator will issue an error message.

**EXAMPLE: Computing the number of years in an ordinary annuity**

Suppose you have a $1,000 ordinary annuity earning an 8% return. How many annual end-of-year $150 withdrawals can be made?

**Answer:**

The number of years in the annuity can be determined by entering the three relevant variables and computing N.

I/Y = 8; PMT = 150; PV = –1,000; CPT → N = 9.9 years

**EXAMPLE: Computing the rate of return for an annuity**

Suppose you have the opportunity to invest $100 at the end of each of the next five years in exchange for $600 at the end of the fifth year. What is the annual rate of return on this investment?

**Answer:**

The rate of return on this investment can be determined by entering the relevant data and solving for I/Y.

N = 5; FV = $600; PMT = –100; CPT → I/Y = 9.13%

**EXAMPLE: Computing the discount rate for an annuity**

What rate of return will you earn on an ordinary annuity that requires a $700 deposit today and promises to pay $100 per year at the end of each of the next 10 years?

**Answer:**

The discount rate on this annuity is determined by entering the three known values and computing I/Y.

N = 10; PV = –700; PMT = 100; CPT → I/Y = 7.07%

## Funding a Future Obligation

There are many TVM applications where it is necessary to determine the size of the deposit(s) that must be made over a specified period in order to meet a future liability, such as setting up a funding program for future college tuition or a retirement program. In most of these applications, the objective is to determine the size of the payment(s) or deposit(s) necessary to meet a particular monetary goal.

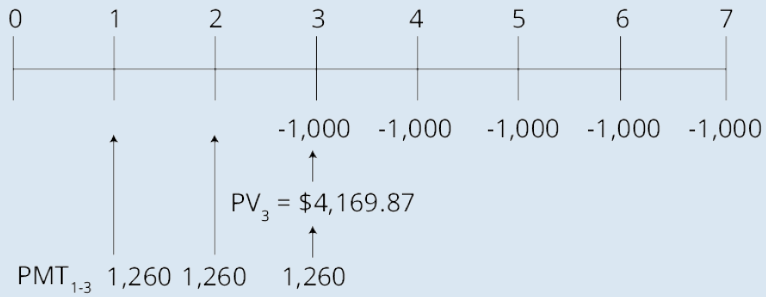
**EXAMPLE: Computing the required payment to fund an annuity due**

Suppose you must make five annual $1,000 payments, the first one starting at the beginning of Year 4 (end of Year 3). To accumulate the money to make these payments, you want to make three equal payments into an investment account, the first to be made one year from today. Assuming a 10% rate of return, what is the amount of these three payments?

**Answer:**

The time line for this annuity problem is shown in the following figure.

**Funding an Annuity Due**




The first step in this type of problem is to determine the amount of money that must be available at the beginning of Year 4 (t = 3) in order to satisfy the payment requirements. This amount is the PV of a 5-year

annuity due at the beginning of Year 4 (end of Year 3). To determine this amount, set your calculator to the BGN mode, enter the relevant data, and compute PV.

N = 5; I/Y = 10; PMT = –1,000; CPT → PV = $PV_3$ = $4,169.87

Alternatively, you can leave your calculator in the END mode, compute the PV of a 5-year ordinary annuity, and multiply by 1.10.

N = 5; I/Y = 10; PMT = –1,000;

CPT → PV = 3,790.79 × 1.1 = $PV_3$ = $4,169.87

A third alternative, with the calculator in END mode, is to calculate the t = 3 value of the last four annuity payments and then add $1,000.

N = 4; I/Y = 10; PMT = –1,000;

CPT → PV = 3,169.87 + 1,000 = $4,169.87 = $PV_3$

$PV_3$ becomes the FV that you need three years from today from your three equal end-of-year deposits. To determine the amount of the three payments necessary to meet this funding requirement, be sure that your calculator is in the END mode, input the relevant data, and compute PMT.

N = 3; I/Y = 10; FV = –4,169.87; CPT → PMT = $1,259.78

The second part of this problem is an ordinary annuity. If you changed your calculator to BGN mode and failed to put it back in the END mode, you will get a PMT of $1,145, which is incorrect.

## The Connection Between Present Values, Future Values, and Series of Cash Flows

As we have explained in the discussion of annuities and series of uneven cash flows, the sum of the present values of the cash flows is the present value of the series. The sum of the future values (at some future time = n) of a series of cash flows is the future value of that series of cash flows.

One interpretation of the present value of a series of cash flows is how much would have to be put in the bank today in order to make these future withdrawals and exhaust the account with the final withdrawal. Let's illustrate this with cash flows of $100 in Year 1, $200 in Year 2, $300 in Year 3, and an assumed interest rate of 10%.

Calculate the present value of these three cash flows as:

$$\frac{100}{1.1} + \frac{200}{1.1^2} + \frac{300}{1.1^3} = \$481.59$$

If we put $481.59 in an account yielding 10%, at the end of the year we would have 481.59 × 1.1 = $529.75. Withdrawing $100 would leave $429.75.

Over the second year, the $429.75 would grow to 429.75 × 1.1 = $472.73. Withdrawing $200 would leave $272.73.

Over the third year, $272.73 would grow to 272.73 × 1.1 = $300, so that the last withdrawal of $300 would empty the account.

The interpretation of the future value of a series of cash flows is straightforward as well. The FV answers the question, "How much would be in an account when the last of a series of deposits is made?" Using the same three cash flows—$100, $200, and $300—and the same interest rate of 10%, we can calculate the future value of the series as:

$100(1.1)^2 + 200(1.1) + 300 = \$641$

This is simply the sum of the t = 3 value of each of the cash flows. Note that the t = 3 value and the t = 0 (present) value of the series are related by the interest rate, $481.59(1.1)^3 = 641$.

The $100 cash flow (deposit) comes at t = 1, so it will earn interest of 10% compounded for two periods (until t = 3). The $200 cash flow (deposit) will earn 10% between t = 2 and t = 3, and the final cash flow (deposit) of $300 is made at t = 3, so $300 is the future (t = 3) value of that cash flow.

We can also look at the future value in terms of how the account grows over time. At t = 1 we deposit $100, so at t = 2 it has grown to $110 and the $200 deposit at t = 2 makes the account balance $310. Over the next period, the $310 grows to 310 × 1.1 = $341 at t = 3, and the addition of the final $300 deposit puts the account balance at $641. This is, of course, the future value we calculated initially.

### PROFESSOR'S NOTE

This last view of the future value of a series of cash flows suggests a quick way to calculate the future value of an uneven cash flow series. The process described previously for the future value of a series of end-of-period payments can be written mathematically as [(100 × 1.1) + 200] × 1.1 + 300 = 641, and this might be a quick way to do some future value problems.

Note that questions on the future value of an *annuity due* refer to the amount in the account one period after the last deposit is made. If the three deposits considered here were made at the beginning of each period (at t = 0, 1, 2) the amount in the account at the end of three years (t = 3) would be 10% higher (i.e., 641 × 1.1 = $705.10).

The **cash flow additivity principle** refers to the fact that present value of any stream of cash flows equals the sum of the present values of the cash flows. There are different applications of this principle in time value of money problems. If we have two series of cash flows, the sum of the present values of the two series is the same as the present values of the two series taken together, adding cash flows that will be paid at the same point in time. We can also divide up a series of cash flows any way we like, and the present value of the "pieces" will equal the present value of the original series.

**EXAMPLE: Additivity principle**

A security will make the following payments at the end of the next four years: $100, $100, $400, and $100. Calculate the present value of these cash flows using the concept of the present value of an annuity when the appropriate discount rate is 10%.

**Answer:**

We can divide the cash flows so that we have:

| t = 1 | t = 2 | t = 3 | t = 4 | |
|---|---|---|---|---|
| 100 | 100 | 100 | 100 | cash flow series #1 |
| 0 | 0 | 300 | 0 | cash flow series #2 |
| $100 | $100 | $400 | $100 | |

The additivity principle tells us that to get the present value of the original series, we can just add the present values of series #1 (a 4-period annuity) and series #2 (a single payment three periods from now).

For the annuity: N = 4; PMT = 100; FV = 0, I/Y = 10; CPT → PV = –$316.99

For the single payment: N = 3; PMT = 0; FV = 300; I/Y = 10; CPT → PV = –$225.39

The sum of these two values is 316.99 + 225.39 = $542.38.

The sum of these two (present) values is identical (except for rounding) to the sum of the present values of the payments of the original series:

$$\frac{100}{1.1} + \frac{100}{1.1^2} + \frac{400}{1.1^3} + \frac{100}{1.1^4} = \$542.38$$

## MODULE QUIZ 6.3

*To best evaluate your performance, enter your quiz answers online.*

1. An analyst estimates that XYZ's earnings will grow from $3.00 a share to $4.50 per share over the next eight years. The rate of growth in XYZ's earnings is *closest* to:
   A. 4.9%.
   B. 5.2%.
   C. 6.7%.
2. If $5,000 is invested in a fund offering a rate of return of 12% per year, approximately how many years will it take for the investment to reach $10,000?
   A. 4 years.
   B. 5 years.
   C. 6 years.
3. An investment is expected to produce the cash flows of $500, $200, and $800 at the end of the next three years. If the required rate of return is 12%, the present value of this investment is *closest* to:
   A. $835.
   B. $1,175.
   C. $1,235.
4. If $10,000 is invested today in an account that earns interest at a rate of 9.5%, what is the value of the equal withdrawals that can be taken out of the account at the end of each of the next five years if the investor plans to deplete the account at the end of the time period?
   A. $2,453.
   B. $2,604.
   C. $2,750.
5. Given an 11% rate of return, the amount that must be put into an investment account at the end of each of the next 10 years in order to accumulate $60,000 to pay for a child's education is *closest* to:
   A. $2,500.
   B. $3,588.
   C. $4,432.
6. An investor will receive an annuity of $4,000 a year for 10 years. The first payment is to be received five years from today. At a 9% discount rate, this annuity's worth today is *closest* to:
   A. $16,684.
   B. $18,186.
   C. $25,671.
7. An investor is looking at a $150,000 home. If 20% must be put down and the balance is financed at 9% over the next 30 years, what is the monthly mortgage payment?
   A. $799.33.
   B. $895.21.
   C. $965.55.

## KEY CONCEPTS

### LOS 6.a

An interest rate can be interpreted as the rate of return required in equilibrium for a particular investment, the discount rate for calculating the present value of future cash flows, or as the opportunity cost of consuming now, rather than saving and investing.

### LOS 6.b

The real risk-free rate is a theoretical rate on a single-period loan when there is no expectation of inflation. Nominal risk-free rate = real risk-free rate + expected inflation rate.

Securities may have several risks, and each increases the required rate of return. These include default risk, liquidity risk, and maturity risk.

The required rate of return on a security = real risk-free rate + expected inflation + default risk premium + liquidity premium + maturity risk premium.

### LOS 6.c

The effective annual rate when there are *m* compounding periods =

$\left(1+\frac{\text{stated annual rate}}{\text{m}}\right)^{\text{m}}-1$. Each dollar invested will grow to

$\left(1+\frac{\text{stated annual rate}}{\text{m}}\right)^{\text{m}}$ in one year.

### LOS 6.d

For non-annual time value of money problems, divide the stated annual interest rate by the number of compounding periods per year, *m*, and multiply the number of years by the number of compounding periods per year.

### LOS 6.e

Future value: $FV = PV(1 + I/Y)^N$

Present value: $PV = FV / (1 + I/Y)^N$

An annuity is a series of equal cash flows that occurs at evenly spaced intervals over time. Ordinary annuity cash flows occur at the end of each time period. Annuity due cash flows occur at the beginning of each time period.

Perpetuities are annuities with infinite lives (perpetual annuities):

$$PV_{\text{perpetuity}} = \frac{PMT}{I/Y}$$

The present (future) value of any series of cash flows is equal to the sum of the present (future) values of the individual cash flows.

### LOS 6.f

Constructing a time line showing future cash flows will help in solving many types of TVM problems. Cash flows occur at the end of the period depicted on the time line. The end of one period is the same as the beginning of the next period. For example, a cash flow at the beginning of Year 3 appears at time t = 2 on the time line.

# ANSWER KEY FOR MODULE QUIZZES

## Module Quiz 6.1

1. **C** Interest rates can be interpreted as required rates of return, discount rates, or opportunity costs of current consumption. A risk premium can be, but is not always, a component of an interest rate. (LOS 6.a, 6.b)

2. **A** Real interest rates are those that have been adjusted for inflation. (LOS 6.b)

3. **C** EAR = $[(1 + (0.18 / 12)]^{12} - 1$ = 19.56% (LOS 6.c)

4. **C** N = 1 × 365 = 365; I/Y = 12 / 365 = 0.0328767; PMT = 0; PV = –5,000; CPT → FV = $5,637.37 (LOS 6.d)

## Module Quiz 6.2

1. **C** N = 15; I/Y = 9; PV = –1,000; PMT = 0; CPT → FV = $3,642.48 (LOS 6.e)

2. **A** N = 7; I/Y = 8; FV = –10,000; PMT = 0; CPT → PV = $5,834.90 (LOS 6.e)

3. **A** N = 20; I/Y = 10; PMT = –50,000; FV = 0; CPT → PV = $425,678.19 (LOS 6.e)

4. **C** This is an annuity due. Switch to BGN mode: N = 15; PMT = –8,000; I/Y = 11; FV = 0; CPT → PV = 63,854.92. Switch back to END mode. (LOS 6.e)

5. **B** The key to this problem is to recognize that it is a 4-year annuity due, so switch to BGN mode: N = 4; PMT = –1,000; PV = 0; I/Y = 12; CPT → FV = 5,352.84. Switch back to END mode. (LOS 6.e)

6. **A** 9 / 0.11 = $81.82 (LOS 6.e)

## Module Quiz 6.3

1. **B** N = 8; PV = –3; FV = 4.50; PMT = 0; CPT → I/Y = 5.1989 (LOS 6.e)

2. **C** PV = –5,000; I/Y = 12; FV = 10,000; PMT = 0; CPT → N = 6.12.

   *Note to HP 12C users: One known problem with the HP 12C is that it does not have the capability to round. In this particular question, you will come up with 7, although the correct answer is 6.1163. CFA Institute is aware of this problem, and hopefully you will not be faced with a situation on exam day where the incorrect solution from the HP is one of the answer choices.* (LOS 6.e)

3. **B** Add up the present values of each single cash flow.

   $PV_1$ = N = 1; FV = –500; I/Y = 12; CPT → PV = 446.43

   $PV_2$ = N = 2; FV = –200; I/Y = 12; CPT → PV = 159.44

   $PV_3$ = N = 3; FV = –800; I/Y = 12; CPT → PV = 569.42

   Hence, 446.43 + 159.44 + 569.42 = $1,175.29. (LOS 6.e)

4. **B** PV = –10,000; I/Y = 9.5; N = 5; FV = 0; CPT → PMT = $2,604.36 (LOS 6.e)

5. **B** N = 10; I/Y = 11; FV = –60,000; PV = 0; CPT → PMT = $3,588.08 (LOS 6.e)

6. **B** Two steps: (1) Find the PV of the 10-year annuity: N = 10; I/Y = 9; PMT = –4,000; FV = 0; CPT → PV = 25,670.63. This is the present value as of the end of Year 4; (2) Discount PV of the annuity back four years: N = 4; PMT = 0; FV = –25,670.63; I/Y = 9; CPT → PV = 18,185.72. (LOS 6.e)

7. **C** N = 30 × 12 = 360; I/Y = 9 / 12 = 0.75; PV = –150,000(1 − 0.2) = –120,000; FV = 0; CPT → PMT = $965.55 (LOS 6.f)

The following is a review of the Quantitative Methods (1) principles designed to address the learning outcome statements set forth by CFA Institute. Cross-Reference to CFA Institute Assigned Reading #7.

# READING 7: STATISTICAL CONCEPTS AND MARKET RETURNS

**Study Session 2**

## EXAM FOCUS

This quantitative review is about the uses of descriptive statistics to summarize and portray important characteristics of large sets of data. The two key areas that you should concentrate on are (1) measures of central tendency and (2) measures of dispersion. Measures of central tendency include the arithmetic mean, geometric mean, weighted mean, median, and mode. Measures of dispersion include the range, mean absolute deviation, and the most important measure for us, variance. These measures quantify the variability of data around its "center." When describing investments, measures of central tendency provide an indication of an investment's expected return. Measures of dispersion indicate the riskiness of an investment. For the Level I exam, you should know the properties of a normal distribution and be able to recognize departures from normality, such as lack of symmetry (skewness) or the extent to which a distribution is peaked (kurtosis).

## MODULE 7.1: DESCRIBING DATA SETS

Video covering this content is available online.

**LOS 7.a: Distinguish between descriptive statistics and inferential statistics, between a population and a sample, and among the types of measurement scales.**

*CFA® Program Curriculum, Volume 1, page 371*

The word **statistics** is used to refer to data (e.g., the average return on XYZ stock was 8% over the last 10 years) and to the methods we use to analyze data. Statistical methods fall into one of two categories, **descriptive statistics** or **inferential statistics**.

*Descriptive statistics* are used to summarize the important characteristics of large data sets. The focus of this topic review is on the use of descriptive statistics to consolidate a mass of numerical data into useful information.

*Inferential statistics*, which will be discussed in subsequent topic reviews, pertain to the procedures used to make forecasts, estimates, or judgments about a large set of data on the basis of the statistical characteristics of a smaller set (a sample).

A **population** is defined as the set of all possible members of a stated group. A cross-section of the returns of all of the stocks traded on the New York Stock Exchange (NYSE) is an example of a population.

It is frequently too costly or time consuming to obtain measurements for every member of a population, if it is even possible. In this case, a sample may be used. A **sample** is defined as a subset of the population of interest. Once a population has been defined, a sample can be

drawn from the population, and the sample's characteristics can be used to describe the population as a whole. For example, a sample of 30 stocks may be selected from among all of the stocks listed on the NYSE to represent the population of all NYSE-traded stocks.

## Types of Measurement Scales

Different statistical methods use different levels of measurement, or measurement scales. **Measurement scales** may be classified into one of four major categories:

- **Nominal scales.** Nominal scales are the level of measurement that contains the least information. Observations are classified or counted with no particular order. An example would be assigning the number 1 to a municipal bond fund, the number 2 to a corporate bond fund, and so on for each fund style.
- **Ordinal scales.** Ordinal scales represent a higher level of measurement than nominal scales. When working with an ordinal scale, every observation is assigned to one of several categories. Then these categories are ordered with respect to a specified characteristic. For example, the ranking of 1,000 small cap growth stocks by performance may be done by assigning the number 1 to the 100 best performing stocks, the number 2 to the next 100 best performing stocks, and so on, assigning the number 10 to the 100 worst performing stocks. Based on this type of measurement, it can be concluded that a stock ranked 3 is better than a stock ranked 4, but the scale reveals nothing about performance differences or whether the difference between a 3 and a 4 is the same as the difference between a 4 and a 5.
- **Interval scale.** Interval scale measurements provide relative ranking, like ordinal scales, plus the assurance that differences between scale values are equal. Temperature measurement in degrees is a prime example. Certainly, 49°C is hotter than 32°C, and the temperature difference between 49°C and 32°C is the same as the difference between 67°C and 50°C. The weakness of the interval scale is that a measurement of zero does not necessarily indicate the total absence of what we are measuring. This means that interval-scale-based ratios are meaningless. For example, 30°F is not three times as hot as 10°F.
- **Ratio scales.** Ratio scales represent the most refined level of measurement. Ratio scales provide ranking and equal differences between scale values, and they also have a true zero point as the origin. Order, intervals, and ratios all make sense with a ratio scale. The measurement of money is a good example. If you have zero dollars, you have no purchasing power, but if you have $4.00, you have twice as much purchasing power as a person with $2.00.

### PROFESSOR'S NOTE

Candidates sometimes use the French word for black, *noir*, to remember the types of scales in order of precision: Nominal, Ordinal, Interval, Ratio.

**LOS 7.b: Define a parameter, a sample statistic, and a frequency distribution.**

*CFA® Program Curriculum, Volume 1, page 371*

A measure used to describe a characteristic of a population is referred to as a **parameter**. While many population parameters exist, investment analysis usually utilizes just a few, particularly the mean return and the standard deviation of returns.

In the same manner that a parameter may be used to describe a characteristic of a population, a **sample statistic** is used to measure a characteristic of a sample.

A **frequency distribution** is a tabular presentation of statistical data that aids the analysis of large data sets. Frequency distributions summarize statistical data by assigning it to specified groups, or intervals. Also, the data employed with a frequency distribution may be measured using any type of measurement scale.

**PROFESSOR'S NOTE**

Intervals are also known as *classes*.

The following procedure describes how to construct a frequency distribution.

*Step 1: Define the intervals*. The first step in building a frequency distribution is to define the intervals to which data measurements (observations) will be assigned. An interval, also referred to as a class, is the set of values that an observation may take on. The range of values for each interval must have a lower and upper limit and be all-inclusive and nonoverlapping. Intervals must be *mutually exclusive* in a way that each observation can be placed in only one interval, and the total set of intervals should cover the total range of values for the entire population. The number of intervals used is an important consideration. If too few intervals are used, the data may be too broadly summarized, and important characteristics may be lost. On the other hand, if too many intervals are used, the data may not be summarized enough.

*Step 2: Tally the observations*. After the intervals have been defined, the observations must be tallied, or assigned to their appropriate interval.

*Step 3: Count the observations*. Having tallied the data set, the number of observations that are assigned to each interval must be counted. The *absolute frequency*, or simply the frequency, is the actual number of observations that fall within a given interval.

**EXAMPLE: Constructing a frequency distribution**

Use the data in Table A to construct a frequency distribution for the returns on Intelco's common stock.

**Table A: Annual Returns for Intelco, Inc. Common Stock**

| | | | |
|---|---|---|---|
| 10.4% | 22.5% | 11.1% | −12.4% |
| 9.8% | 17.0% | 2.8% | 8.4% |
| 34.6% | −28.6% | 0.6% | 5.0% |
| −17.6% | 5.6% | 8.9% | 40.4% |
| −1.0% | −4.2% | −5.2% | 21.0% |

**Answer:**

*Step 1: Defining the interval*. For Intelco's stock, the range of returns is 69.0% (–28.6% → 40.4%). Using a return interval of 1% would result in 69 separate intervals, which in this case is too many. So let's use eight nonoverlapping intervals with a width of 10%. The lowest return intervals will be $-30\% \le R_t < -20\%$, and the intervals will increase to $40\% \le R_t \le 50\%$.

*Step 2: Tally the observations and count the observations within each interval*. The tallies and counts of the observations are presented in Table B.

**Table B: Tally and Interval Count for Returns Data**

| Intervals | Tallies | Absolute Frequency |
|---|---|---|
| $-30\% \le R_t < -20\%$ | / | 1 |

| | | |
|---|---|---|
| $-20\% \le R_t < -10\%$ | // | 2 |
| $-10\% \le R_t < 0\%$ | /// | 3 |
| $0\% \le R_t < 10\%$ | /////// | 7 |
| $10\% \le R_t < 20\%$ | /// | 3 |
| $20\% \le R_t < 30\%$ | // | 2 |
| $30\% \le R_t < 40\%$ | / | 1 |
| $40\% \le R_t < 50\%$ | / | 1 |
| Total | | 20 |

Tallying and counting the observations generates a frequency distribution that summarizes the pattern of annual returns on Intelco common stock. Notice that the interval with the greatest (absolute) frequency is the ($0\% \le R_t < 10\%$) interval, which includes seven return observations. For any frequency distribution, the interval with the greatest frequency is referred to as the **modal interval**.

## LOS 7.c: Calculate and interpret relative frequencies and cumulative relative frequencies, given a frequency distribution.

*CFA® Program Curriculum, Volume 1, page 373*

The **relative frequency** is another useful way to present data. The relative frequency is calculated by dividing the absolute frequency of each return interval by the total number of observations. Simply stated, relative frequency is the percentage of total observations falling within each interval. Continuing with our example, the relative frequencies are presented in Figure 7.1.

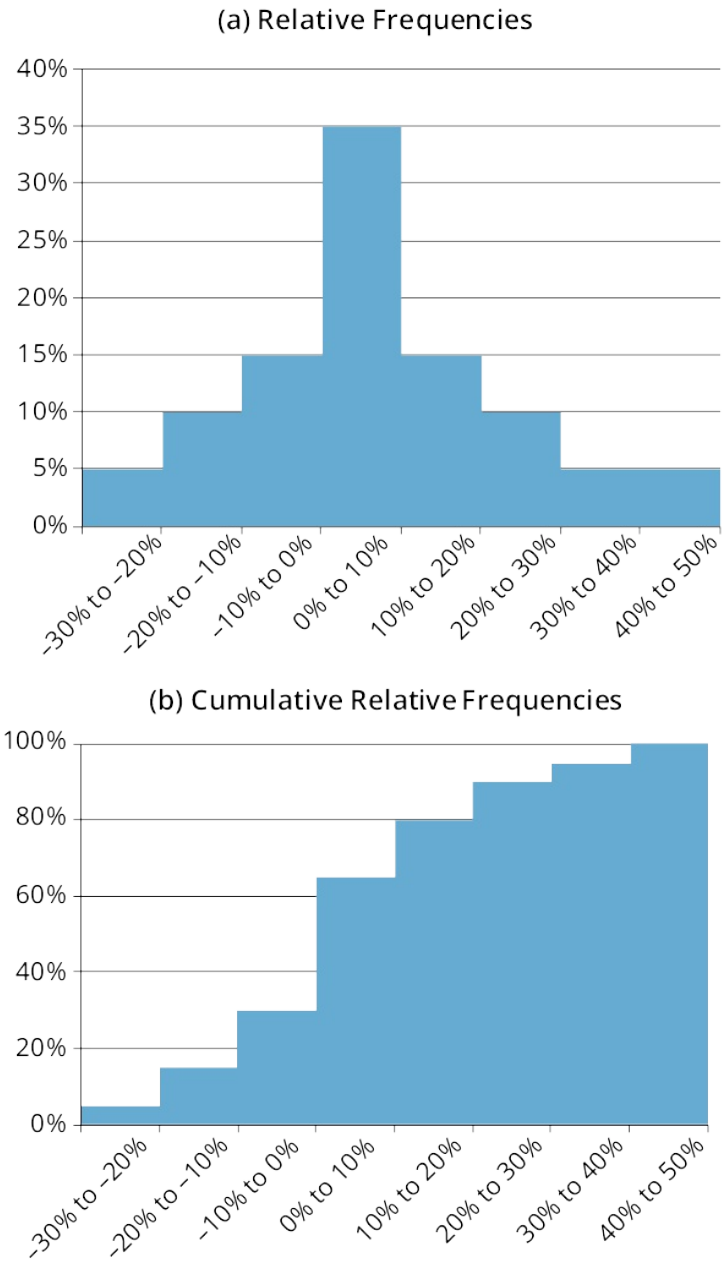
**Figure 7.1: Absolute and Relative Frequencies of Intelco Returns**

| Interval | Absolute Frequency | Relative Frequency |
|---|---|---|
| $-30\% \le R_t < -20\%$ | 1 | 1/20 = 0.05, or 5% |
| $-20\% \le R_t < -10\%$ | 2 | 2/20 = 0.10, or 10% |
| $-10\% \le R_t < 0\%$ | 3 | 3/20 = 0.15, or 15% |
| $0\% \le R_t < 10\%$ | 7 | 7/20 = 0.35, or 35% |
| $10\% \le R_t < 20\%$ | 3 | 3/20 = 0.15, or 15% |
| $20\% \le R_t < 30\%$ | 2 | 2/20 = 0.10, or 10% |
| $30\% \le R_t < 40\%$ | 1 | 1/20 = 0.05, or 5% |
| $40\% \le R_t \le 50\%$ | 1 | 1/20 = 0.05, or 5% |
| Total | 20 | 100% |

It is also possible to compute the **cumulative absolute frequency** and **cumulative relative frequency** by summing the absolute or relative frequencies starting at the lowest interval and progressing through the highest. The relative and cumulative relative frequencies for the Intelco stock returns example are presented in Figure 7.2.

The cumulative absolute frequency or cumulative relative frequency for any given interval is the sum of the absolute or relative frequencies up to and including the given interval. For example, the cumulative absolute frequency for $R_t < 10\%$ is 13 = 1 + 2 + 3 + 7 and the cumulative relative frequency over this range is 5% + 10% + 15% + 35% = 65%.

**Figure 7.2: Relative and Cumulative Frequencies of Intelco Returns**



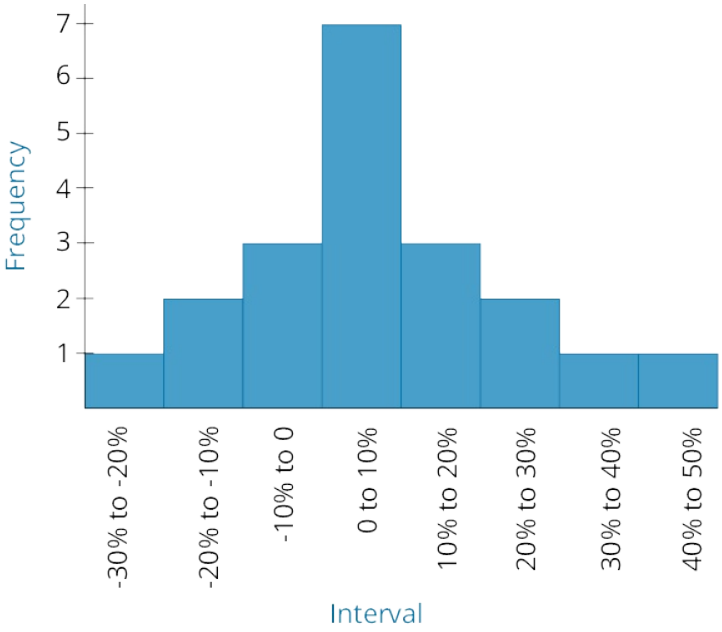

## LOS 7.d: Describe the properties of a data set presented as a histogram or a frequency polygon.

*CFA® Program Curriculum, Volume 1, page 381*

A **histogram** is the graphical presentation of the absolute frequency distribution. A histogram is simply a bar chart of continuous data that has been classified into a frequency distribution. The attractive feature of a histogram is that it allows us to quickly see where most of the observations are concentrated.

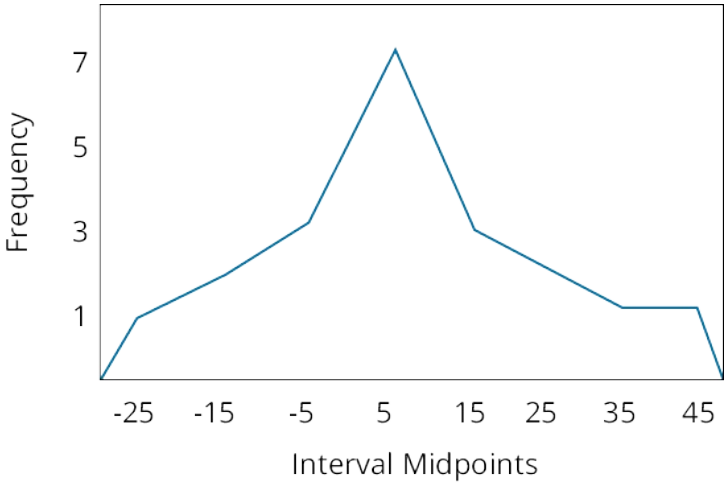
To construct a histogram, the intervals are scaled on the horizontal axis and the absolute frequencies are scaled on the vertical axis. The histogram for the Intelco returns data in Table B from the previous example is provided in Figure 7.3.

**Figure 7.3: Histogram of Intelco Stock Return Data**




To construct a **frequency polygon**, the midpoint of each interval is plotted on the horizontal axis, and the absolute frequency for that interval is plotted on the vertical axis. Each point is then connected with a straight line. The frequency polygon for the returns data used in our example is illustrated in Figure 7.4.

**Figure 7.4: Frequency Polygon of Intelco Stock Return Data**




## MODULE QUIZ 7.1

*To best evaluate your performance, enter your quiz answers online.*

1. In the card game of poker, winning hands are determined by rank (high card, pair, two pair, etc.). The scale of poker hand rankings is:
   A. a nominal scale.
   B. an ordinal scale.
   C. an interval scale.
2. The intervals in a frequency distribution should always have which of the following characteristics? The intervals should always:
   A. be truncated.
   B. be open ended.

C. be nonoverlapping.

**Use the following frequency distribution for Questions 3 through 5.**

| Return, R | Frequency |
|---|---|
| –10% up to 0% | 3 |
| 0% up to 10% | 7 |
| 10% up to 20% | 3 |
| 20% up to 30% | 2 |
| 30% up to 40% | 1 |

3. The number of intervals in this frequency table is:
   A. 1.
   B. 5.
   C. 16.
4. The sample size is:
   A. 1.
   B. 5.
   C. 16.
5. The relative frequency of the second interval is:
   A. 10.0%.
   B. 16.0%.
   C. 43.8%.
6. The vertical axis of a histogram shows:
   A. the frequency with which observations occur.
   B. the range of observations within each interval.
   C. the intervals into which the observations are arranged.

## MODULE 7.2: MEANS AND VARIANCE

Video covering this content is available online.

**LOS 7.e: Calculate and interpret measures of central tendency, including the population mean, sample mean, arithmetic mean, weighted average or mean, geometric mean, harmonic mean, median, and mode.**

*CFA® Program Curriculum, Volume 1, page 386*

**Measures of central tendency** identify the center, or average, of a data set. This central point can then be used to represent the typical, or expected, value in the data set.

To compute the **population mean**, all the observed values in the population are summed (ΣX) and divided by the number of observations in the population, *N*. Note that the population mean is unique in that a given population only has one mean. The population mean is expressed as:

$$\mu = \frac{\sum_{i=1}^{N} X_i}{N}$$

The **sample mean** is the sum of all the values in a sample of a population, ΣX, divided by the number of observations in the sample, *n*. It is used to make *inferences* about the population mean. The sample mean is expressed as:

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

Note the use of *n*, the sample size, versus *N*, the population size.

> **EXAMPLE: Population mean and sample mean**
>
> You have calculated the stock returns for AXZ Corporation over the last five years as 25%, 34%, 19%, 54%, and 17%. Given this information, estimate the mean of the distribution of returns.
>
> **Answer:**
>
> The sample mean can be used as an estimate of the mean of the distribution:
>
> $$\overline{X} = \text{ sample mean } = \frac{25+34+19+54+17}{5} = 29.8\%$$

The population mean and sample mean are both examples of **arithmetic means**. The arithmetic mean is the sum of the observation values divided by the number of observations. It is the most widely used measure of central tendency and has the following properties:

- All interval and ratio data sets have an arithmetic mean.
- All data values are considered and included in the arithmetic mean computation.
- A data set has only one arithmetic mean (i.e., the arithmetic mean is unique).
- The sum of the deviations of each observation in the data set from the mean is always zero.

The arithmetic mean is the only measure of central tendency for which the sum of the deviations from the mean is zero. Mathematically, this property can be expressed as follows:

$$\text{sum of mean deviations} = \sum_{i=1}^{n} (X_i - \overline{X}) = 0$$

Unusually large or small values can have a disproportionate effect on the computed value for the arithmetic mean. The mean of 1, 2, 3, and 50 is 14 and is not a good indication of what the individual data values really are. On the positive side, the arithmetic mean uses all the information available about the observations. The arithmetic mean of a sample from a population is the best estimate of both the true mean of the sample and the value of the next observation.

The computation of a **weighted mean** recognizes that different observations may have a disproportionate influence on the mean. The weighted mean of a set of numbers is computed with the following equation:

$$\overline{X}_W = \sum_{i=1}^{n} w_i X_i = (w_1 X_1 + w_2 X_2 + \ldots + w_n X_n)$$

where:

$X_1, X_2, \ldots X_n$ = observed values

$w_1, w_2, \ldots w_n$ = corresponding weights associated with each of the observations such that $\Sigma w_i = 1$

**EXAMPLE: Weighted mean as a portfolio return**

A portfolio consists of 50% common stocks, 40% bonds, and 10% cash. If the return on common stocks is 12%, the return on bonds is 7%, and the return on cash is 3%, what is the portfolio return?

**Answer:**

$$\overline{X}_W = w_{stock} R_{stock} + w_{bonds} R_{bonds} + w_{cash} R_{cash}$$

$$\overline{X}_W = (0.50 \times 0.12) + (0.40 \times 0.07) + (0.10 \times 0.03) = 0.091, \text{ or } 9.1\%$$

The example illustrates an extremely important investments concept: *the return for a portfolio is the weighted average of the returns of the individual assets in the portfolio*. Asset weights are market weights, the market value of each asset relative to the market value of the entire portfolio.

The **median** is the midpoint of a data set when the data is arranged in ascending or descending order. Half the observations lie above the median and half are below. To determine the median, arrange the data from the highest to the lowest value, or lowest to highest value, and find the middle observation.

The median is important because the arithmetic mean can be affected by extremely large or small values (outliers). When this occurs, the median is a better measure of central tendency than the mean because it is not affected by extreme values that may actually be the result of errors in the data.

**EXAMPLE: The median using an odd number of observations**

What is the median return for five portfolio managers with 10-year annualized total returns record of: 30%, 15%, 25%, 21%, and 23%?

**Answer:**

First, arrange the returns in descending order.

30%, 25%, 23%, 21%, 15%

Then, select the observation that has an equal number of observations above and below it—the one in the middle. For the given data set, the third observation, 23%, is the median value.

**EXAMPLE: The median using an even number of observations**

Suppose we add a sixth manager to the previous example with a return of 28%. What is the median return?

**Answer:**

Arranging the returns in descending order gives us:

30%, 28%, 25%, 23%, 21%, 15%

With an even number of observations, there is no single middle value. The median value in this case is the arithmetic mean of the two middle observations, 25% and 23%. Thus, the median return for the six managers is 24.0% = 0.5(25 + 23).

Consider that while we calculated the mean of 1, 2, 3, and 50 as 14, the median is 2.5. If the data were 1, 2, 3, and 4 instead, the arithmetic mean and median would both be 2.5.

The **mode** is the value that occurs most frequently in a data set. A data set may have more than one mode or even no mode. When a distribution has one value that appears most frequently, it is said to be **unimodal**. When a set of data has two or three values that occur most frequently, it is said to be **bimodal** or **trimodal**, respectively.

**EXAMPLE: The mode**

What is the mode of the following data set?

Data set: [30%, 28%, 25%, 23%, 28%, 15%, 5%]

**Answer:**

The mode is 28% because it is the value appearing most frequently.

The **geometric mean** is often used when calculating investment returns over multiple periods or when measuring compound growth rates. The general formula for the geometric mean, *G*, is as follows:

$$G = \sqrt[n]{X_1 \times X_2 \times \ldots \times X_n} = (X_1 \times X_2 \times \ldots \times X_n)^{1/n}$$

Note that this equation has a solution only if the product under the radical sign is non-negative.

When calculating the geometric mean for a returns data set, it is necessary to add 1 to each value under the radical and then subtract 1 from the result. The geometric mean return ($R_G$) can be computed using the following equation:

$$1 + R_G = \sqrt[n]{(1 + R_1) \times (1 + R_2) \times \ldots \times (1 + R_n)}$$

where:
$R_t$ = the return for period t

**EXAMPLE: Geometric mean return**

For the last three years, the returns for Acme Corporation common stock have been –9.34%, 23.45%, and 8.92%. Compute the compound annual rate of return over the 3-year period.

**Answer:**

$$\begin{aligned} 1 + R_G &= \sqrt[3]{(1 - 0.0934) \times (1 + 0.2345) \times (1 + 0.0892)} \\ 1 + R_G &= \sqrt[3]{0.9066 \times 1.2345 \times 1.0892} = \sqrt[3]{1.21903} \\ &= (1.21903)^{1/3} = 1.06825 \\ R_G &= 1.06825 - 1 = 6.825\% \end{aligned}$$

Solve this type of problem with your calculator as follows:

- On the TI, enter 1.21903 [$y^X$] 3 [1/x] [=]
- On the HP, enter 1.21903 [ENTER] 3 [1/x] [$y^X$]

## PROFESSOR'S NOTE

The geometric mean is always less than or equal to the arithmetic mean, and the difference increases as the dispersion of the observations increases. The only time the arithmetic and geometric means are equal is when there is no variability in the observations (i.e., all observations are equal).

A **harmonic mean** is used for certain computations, such as the average cost of shares purchased over time. The harmonic mean is calculated as $\frac{N}{\sum_{i=1}^{N} \frac{1}{X_i}}$, where there are *N* values of $X_i$.

> **EXAMPLE: Calculating average cost with the harmonic mean**
>
> An investor purchases $1,000 of mutual fund shares each month, and over the last three months the prices paid per share were $8, $9, and $10. What is the average cost per share?
>
> **Answer:**
>
> $$\overline{X}_H = \frac{3}{1/8 + 1/9 + 1/10} = \$8.926 \text{ per share}$$
>
> To check this result, calculate the total shares purchased as
>
> $\frac{1{,}000}{8} + \frac{1{,}000}{9} + \frac{1{,}000}{10} = 336.11$ shares.
>
> The average price is $\frac{\$3{,}000}{336.11} = \$8.926$ per share.

The previous example illustrates the interpretation of the harmonic mean in its most common application. Note that the average price paid per share ($8.93) is less than the arithmetic average of the share prices, $\frac{8+9+10}{3} = 9$.

For values that are not all equal: harmonic mean < geometric mean < arithmetic mean. This mathematical fact is the basis for the claimed benefit of purchasing the same dollar amount of mutual fund shares each month or each week. Some refer to this practice as "dollar cost averaging."

### LOS 7.f: Calculate and interpret quartiles, quintiles, deciles, and percentiles.

*CFA® Program Curriculum, Volume 1, page 403*

**Quantile** is the general term for a value at or below which a stated proportion of the data in a distribution lies. Examples of quantiles include:

- *Quartiles*—the distribution is divided into quarters.
- *Quintile*—the distribution is divided into fifths.
- *Decile*—the distribution is divided into tenths.
- *Percentile*—the distribution is divided into hundredths (percents).

Note that any quantile may be expressed as a percentile. For example, the third quartile partitions the distribution at a value such that three-fourths, or 75%, of the observations fall below that value. Thus, the third quartile is the 75th percentile.

The formula for the position of the observation at a given percentile, *y*, with *n* data points sorted in ascending order is:

$$L_y = (n+1)\frac{y}{100}$$

Quantiles and measures of central tendency are known collectively as **measures of location**.

**EXAMPLE: Quartiles**

What is the third quartile for the following distribution of returns?

8%, 10%, 12%, 13%, 15%, 17%, 17%, 18%, 19%, 23%

**Answer:**

The third quartile is the point below which 75% of the observations lie. Recognizing that there are 10 observations in the data set, the third quartile can be identified as:

$$L_y = (10 + 1) \times \frac{75}{100} = 8.25$$

When the data is arranged in ascending order, the third quartile is one-fourth (0.25) of the way from the eighth data point (18%) to the ninth data point (19%), or 18.25%. This means that 75% of all observations lie below 18.25%.

## LOS 7.g: Calculate and interpret 1) a range and a mean absolute deviation and 2) the variance and standard deviation of a population and of a sample.

*CFA® Program Curriculum, Volume 1, page 411*

*Dispersion* is defined as the *variability around the central tendency*. The common theme in finance and investments is the tradeoff between reward and variability, where the central tendency is the measure of the reward and dispersion is a measure of risk.

The **range** is a relatively simple measure of variability, but when used with other measures it provides extremely useful information. The range is the distance between the largest and the smallest value in the data set, or:

range = maximum value − minimum value

**EXAMPLE: The range**

What is the range for the 5-year annualized total returns for five investment managers if the managers' individual returns were 30%, 12%, 25%, 20%, and 23%?

**Answer:**

range = 30 − 12 = 18%

The **mean absolute deviation** (MAD) is the average of the absolute values of the deviations of individual observations from the arithmetic mean.

$$\text{MAD} = \frac{\sum_{i=1}^{n} |X_i - \bar{X}|}{n}$$

The computation of the MAD uses the absolute values of each deviation from the mean because the sum of the actual deviations from the arithmetic mean is zero.

**EXAMPLE: MAD**

What is the MAD of the investment returns for the five managers discussed in the preceding example? How is it interpreted?

**Answer:**

annualized returns: [30%, 12%, 25%, 20%, 23%]

$$\overline{X} = \frac{[30+12+25+20+23]}{5} = 22\%$$

$$\text{MAD} = \frac{[|30-22|+|12-22|+|25-22|+|20-22|+|23-22|]}{5}$$

$$\text{MAD} = \frac{[8+10+3+2+1]}{5} = 4.8\%$$

This result can be interpreted to mean that, on average, an individual return will deviate ±4.8% from the mean return of 22%.

The **population variance** is defined as the average of the squared deviations from the mean. The population variance ($\sigma^2$) uses the values for all members of a population and is calculated using the following formula:

$$\sigma^2 = \frac{\sum_{i=1}^{N} (X_i - \mu)2}{N}$$

**EXAMPLE: Population variance, $\sigma^2$**

Assume the 5-year annualized total returns for the five investment managers used in the earlier example represent *all* of the managers at a small investment firm. What is the population variance of returns?

**Answer:**

$$\mu = \frac{[30+12+25+20+23]}{5} = 22\%$$

$$\sigma^2 = \frac{\left[(30-22)^2+(12-22)^2+(25-22)^2+(20-22)^2+(23-22)^2\right]}{5}$$

$$= 35.60\left(\%^2\right)$$

Interpreting this result, we can say that the average variation from the mean return is 35.60% squared. Had we done the calculation using decimals instead of whole percents, the variance would be 0.00356. What is a percent squared? Yes, this is nonsense, but let's see what we can do so that it makes more sense.

As you have just seen, a major problem with using the variance is the difficulty of interpreting it. The computed variance, unlike the mean, is in terms of squared units of measurement. How does one interpret squared percents, squared dollars, or squared yen? This problem is mitigated through the use of the *standard deviation*. The **population standard deviation**, $\sigma$, is the square root of the population variance and is calculated as follows:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} (X-\mu)^2}{N}}$$

**EXAMPLE: Population standard deviation, $\sigma$**

Using the data from the preceding examples, compute the population standard deviation.

**Answer:**

$$\sigma = \sqrt{\frac{(30-22)^2+(12-22)^2+(25-22)^2+(20-22)^2+(23-22)^2}{5}} = \sqrt{35.60} = 5.97\%$$

Calculated with decimals instead of whole percents, we would get:

$$\sigma^2 = 0.00356 \text{ and } \sigma = \sqrt{0.00356} = 0.05966 = 5.97\%$$

Since the population standard deviation and population mean are both expressed in the same units (percent), these values are easy to relate. The outcome of this example indicates that the mean return is 22% and the standard deviation about the mean is 5.97%. Note that this is greater than the MAD of 4.8%, a result (σ > MAD) that holds in general.

The **sample variance**, $s^2$, is the measure of dispersion that applies when we are evaluating a sample of *n* observations from a population. The sample variance is calculated using the following formula:

$$s^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1}$$

In our example for population variance, we defined the population as the returns generated by the five managers at a firm over a specific five-year period. We calculated the standard deviation of those managers' returns for the specific five-year period. If there were 10 managers at the firm, the five returns we used would be a sample of manager returns for that firm.

In practice, we seldom have historical returns data that comprise an entire population. Consider the returns for the S&P 500 index over the last 50 years. To estimate the standard deviation of returns on the index, we would calculate the sample standard deviation, because 50 years of returns are a sample of all the possible returns outcomes for the index. The returns would be a population if we were asked to calculate the (population) standard deviation of S&P 500 returns over the specific 50-year period, rather than to estimate the standard deviation of S&P 500 returns in general.

The most noteworthy difference from the formula for population variance is that the denominator for $s^2$ is n − 1, one less than the sample size *n*, where $\sigma^2$ uses the entire population size *N*. Another difference is the use of the sample mean, $\overline{X}$, instead of the population mean, μ. Based on the mathematical theory behind statistical procedures, the use of the entire number of sample observations, *n*, instead of n − 1 as the divisor in the computation of $s^2$, will systematically *underestimate* the population parameter, $\sigma^2$, particularly for small sample sizes. This systematic underestimation causes the sample variance to be what is referred to as a **biased estimator** of the population variance. Using n − 1 instead of *n* in the denominator, however, improves the statistical properties of $s^2$ as an estimator of $\sigma^2$. Thus, $s^2$, as expressed in the previous equation, is considered to be an unbiased estimator of $\sigma^2$.

**EXAMPLE: Sample variance**

Assume that the 5-year annualized total returns for the five investment managers used in the preceding examples represent only a sample of the managers at a large investment firm. What is the sample variance of these returns?

**Answer:**

$$\overline{X} = \frac{[30+12+25+20+23]}{5} = 22\%$$

$$s^2 = \frac{\left[(30-22)^2+(12-22)^2+(25-22)^2+(20-22)^2+(23-22)^2\right]}{5-1} = 44.5\ (\%^2)$$

Thus, the sample variance of 44.5($\%^2$) can be interpreted to be an unbiased estimator of the population variance. Note that 44.5 "percent squared" is 0.00445 and you will get this value if you put the percent

returns in decimal form [e.g., $(0.30 - 0.22)^2$, etc.].

As with the population standard deviation, the **sample standard deviation** can be calculated by taking the square root of the sample variance. The sample standard deviation, *s*, is defined as:

$$s = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1}}$$

**EXAMPLE: Sample standard deviation**

Compute the sample standard deviation based on the result of the preceding example.

**Answer:**

Since the sample variance for the preceding example was computed to be $44.5(\%^2)$, the sample standard deviation is:

$$s = [44.5(\%^2)]^{1/2} = 6.67\% \text{ or } \sqrt{0.00445} = 0.0667$$

The results shown here mean that the sample standard deviation, s = 6.67%, can be interpreted as an unbiased estimator of the population standard deviation, σ.

## MODULE QUIZ 7.2

*To best evaluate your performance, enter your quiz answers online.*

**Use the following data to answer Questions 1 through 9.**

**XYZ Corp. Annual Stock Returns**

| 2013 | 2014 | 2015 | 2016 | 2017 | 2018 |
|---|---|---|---|---|---|
| 22% | 5% | –7% | 11% | 2% | 11% |

1. What is the arithmetic mean return for XYZ stock?
   A. 7.3%.
   B. 8.0%.
   C. 11.0%.
2. What is the median return for XYZ stock?
   A. 7.3%.
   B. 8.0%.
   C. 11.0%.
3. What is the mode of the returns for XYZ stock?
   A. 7.3%.
   B. 8.0%.
   C. 11.0%.
4. What is the range for XYZ stock returns?
   A. 11.0%.
   B. 22.0%.
   C. 29.0%.
5. What is the mean absolute deviation for XYZ stock returns?
   A. 5.20%.
   B. 7.33%.
   C. 29.0%.

6. Assuming that the distribution of XYZ stock returns is a population, what is the population variance?
   A. $6.8\%^2$.
   B. $7.7\%^2$.
   C. $80.2\%^2$.
7. Assuming that the distribution of XYZ stock returns is a population, what is the population standard deviation?
   A. 5.02%.
   B. 8.96%.
   C. 46.22%.
8. Assuming that the distribution of XYZ stock returns is a sample, the sample variance is *closest* to:
   A. $5.0\%^2$.
   B. $72.4\%^2$.
   C. $96.3\%^2$.
9. Assuming that the distribution of XYZ stock returns is a sample, what is the sample standard deviation?
   A. 9.8%.
   B. 72.4%.
   C. 96.3%.

**Use the following data to answer Questions 10 and 11.**

The annual returns for FJW's common stock over the years 2013, 2014, 2015, and 2016 were 15%, 19%, –8%, and 14%.

10. What is the arithmetic mean return for FJW's common stock?
    A. 10.00%.
    B. 14.00%.
    C. 15.25%.
11. What is the geometric mean return for FJW's common stock?
    A. 9.45%.
    B. 14.21%.
    C. It cannot be determined because one of the returns is negative.
12. The harmonic mean of 3, 4, and 5 is:
    A. 3.74.
    B. 3.83.
    C. 4.12.
13. Given the following observations:

    2, 4, 5, 6, 7, 9, 10, 11

    The 65th percentile is *closest* to:
    A. 5.85.
    B. 6.55.
    C. 8.70.

# MODULE 7.3: SKEW AND KURTOSIS

Video covering this content is available online.

## LOS 7.h: Calculate and interpret the proportion of observations falling within a specified number of standard deviations of the mean using Chebyshev's inequality.

*CFA® Program Curriculum, Volume 1, page 420*

**Chebyshev's inequality** states that for any set of observations, whether sample or population data and regardless of the shape of the distribution, the percentage of the observations that lie within *k* standard deviations of the mean is *at least* $1 - 1/k^2$ for all $k > 1$.

> **EXAMPLE: Chebyshev's inequality**
>
> What is the minimum percentage of any distribution that will lie within ±2 standard deviations of the mean?
>
> **Answer:**
>
> Applying Chebyshev's inequality, we have:
>
> $$1 - 1/k^2 = 1 - 1/2^2 = 1 - 1/4 = 0.75 \text{ or } 75\%$$

According to Chebyshev's inequality, the following relationships hold for any distribution. At least:

- 36% of observations lie within ±1.25 standard deviations of the mean.
- 56% of observations lie within ±1.50 standard deviations of the mean.
- 75% of observations lie within ±2 standard deviations of the mean.
- 89% of observations lie within ±3 standard deviations of the mean.
- 94% of observations lie within ±4 standard deviations of the mean.

The importance of Chebyshev's inequality is that it applies to any distribution. If we actually know the underlying distribution is normal, for example, we can be even more precise about the percentage of observations that will fall within 2 or 3 standard deviations of the mean.

### LOS 7.i: Calculate and interpret the coefficient of variation.

*CFA® Program Curriculum, Volume 1, page 422*

A direct comparison between two or more measures of dispersion may be difficult. For instance, suppose you are comparing the annual returns distribution for retail stocks with a mean of 8% and an annual returns distribution for a real estate portfolio with a mean of 16%. A direct comparison between the dispersion of the two distributions is not meaningful because of the relatively large difference in their means. To make a meaningful comparison, a relative measure of dispersion must be used. **Relative dispersion** is the amount of variability in a distribution relative to a reference point or benchmark. Relative dispersion is commonly measured with the **coefficient of variation** (CV), which is computed as:

$$\text{CV} = \frac{s_x}{\bar{X}} = \frac{\text{standard deviation of x}}{\text{average value of x}}$$

CV measures the amount of dispersion in a distribution relative to the distribution's mean. It is useful because it enables us to make a direct comparison of dispersion across different sets of data. In an investments setting, the CV is used to measure the risk (variability) per unit of expected return (mean).

> **EXAMPLE: Coefficient of variation**
>
> You have just been presented with a report that indicates that the mean monthly return on T-bills is 0.25% with a standard deviation of 0.36%, and the mean monthly return for the S&P 500 is 1.09% with a standard

deviation of 7.30%. Your unit manager has asked you to compute the CV for these two investments and to interpret your results.

**Answer:**

$$CV_{\text{T-bills}} = \frac{0.36}{0.25} = 1.44$$

$$CV_{\text{S\&P 500}} = \frac{7.30}{1.09} = 6.70$$

These results indicate that there is less dispersion (risk) per unit of monthly return for T-bills than there is for the S&P 500 (1.44 versus 6.70).

### PROFESSOR'S NOTE

To remember the formula for CV, remember that the coefficient of variation is a measure of variation, so standard deviation goes in the numerator. CV is variation per unit of return.

## LOS 7.j: Explain skewness and the meaning of a positively or negatively skewed return distribution.

*CFA® Program Curriculum, Volume 1, page 424*

A distribution is **symmetrical** if it is shaped identically on both sides of its mean. Distributional symmetry implies that intervals of losses and gains will exhibit the same frequency. For example, a symmetrical distribution with a mean return of zero will have losses in the –6% to –4% interval as frequently as it will have gains in the +4% to +6% interval. The extent to which a returns distribution is symmetrical is important because the degree of symmetry tells analysts if deviations from the mean are more likely to be positive or negative.

**Skewness**, or skew, refers to the extent to which a distribution is not symmetrical. Nonsymmetrical distributions may be either positively or negatively skewed and result from the occurrence of outliers in the data set. **Outliers** are observations with extraordinarily large values, either positive or negative.

- A *positively skewed* distribution is characterized by many outliers in the upper region, or right tail. A positively skewed distribution is said to be skewed right because of its relatively long upper (right) tail.
- A *negatively skewed* distribution has a disproportionately large amount of outliers that fall within its lower (left) tail. A negatively skewed distribution is said to be skewed left because of its long lower tail.

## LOS 7.k: Describe the relative locations of the mean, median, and mode for a unimodal, nonsymmetrical distribution.

*CFA® Program Curriculum, Volume 1, page 424*

Skewness affects the location of the mean, median, and mode of a distribution.

- For a symmetrical distribution, the mean, median, and mode are equal.
- For a positively skewed, unimodal distribution, the mode is less than the median, which is less than the mean. The mean is affected by outliers; in a positively skewed distribution, there are large, positive outliers which will tend to "pull" the mean upward, or more positive. An example of a positively skewed distribution is that of

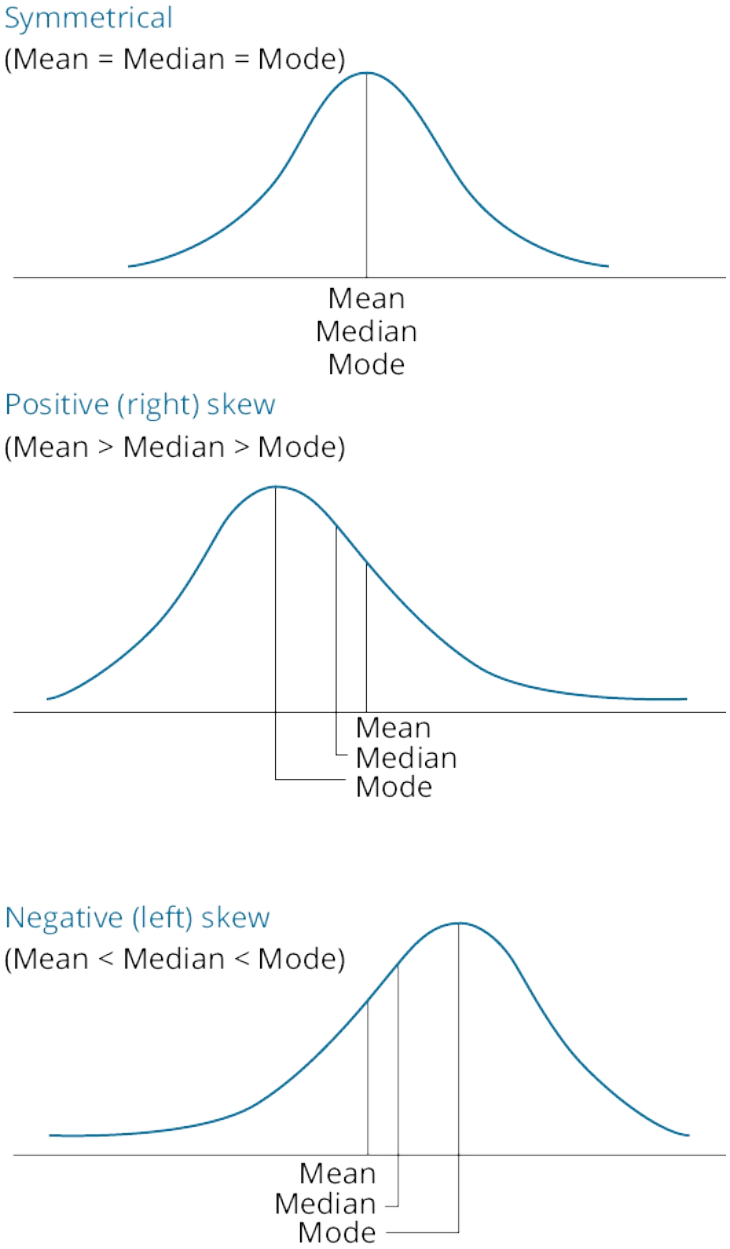
housing prices. Suppose that you live in a neighborhood with 100 homes; 99 of them sell for $100,000, and one sells for $1,000,000. The median and the mode will be $100,000, but the mean will be $109,000. Hence, the mean has been "pulled" upward (to the right) by the existence of one home (outlier) in the neighborhood.

- For a negatively skewed, unimodal distribution, the mean is less than the median, which is less than the mode. In this case, there are large, negative outliers that tend to "pull" the mean downward (to the left).

### PROFESSOR'S NOTE

The key to remembering how measures of central tendency are affected by skewed data is to recognize that skew affects the mean more than the median and mode, and the mean is "pulled" in the direction of the skew. The relative location of the mean, median, and mode for different distribution shapes is shown in Figure 7.5. Note the median is between the other two measures for positively or negatively skewed distributions.

**Figure 7.5: Effect of Skewness on Mean, Median, and Mode**



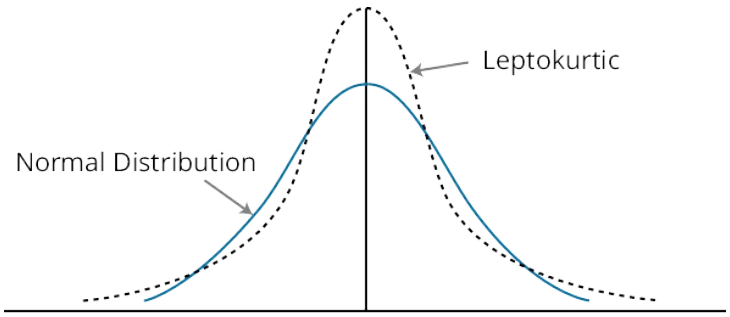
### LOS 7.l: Explain measures of sample skewness and kurtosis.

*CFA® Program Curriculum, Volume 1, page 424*

**Kurtosis** is a measure of the degree to which a distribution is more or less "peaked" than a normal distribution. **Leptokurtic** describes a distribution that is more peaked than a normal distribution, whereas **platykurtic** refers to a distribution that is less peaked, or flatter than a normal distribution. A distribution is **mesokurtic** if it has the same kurtosis as a normal distribution.

As indicated in Figure 7.6, a leptokurtic return distribution will have more returns clustered around the mean and more returns with large deviations from the mean (fatter tails). Relative to a normal distribution, a leptokurtic distribution will have a greater percentage of small deviations from the mean and a greater percentage of extremely large deviations from the mean. This means that there is a relatively greater probability of an observed value being either close to the mean or far from the mean. With regard to an investment returns distribution, a greater likelihood of a large deviation from the mean return is often perceived as an increase in risk.

**Figure 7.6: Kurtosis**




A distribution is said to exhibit **excess kurtosis** if it has either more or less kurtosis than the normal distribution. The computed kurtosis for all normal distributions is three. Statisticians, however, sometimes report excess kurtosis, which is defined as kurtosis minus three. Thus, a normal distribution has excess kurtosis equal to zero, a leptokurtic distribution has excess kurtosis greater than zero, and platykurtic distributions will have excess kurtosis less than zero.

Kurtosis is critical in a risk management setting. Most research about the distribution of securities returns has shown that returns are not normally distributed. Actual securities returns tend to exhibit both skewness and kurtosis. Skewness and kurtosis are critical concepts for risk management because when securities returns are modeled using an assumed normal distribution, the predictions from the models will not take into account the potential for extremely large, negative outcomes. In fact, most risk managers put very little emphasis on the mean and standard deviation of a distribution and focus more on the distribution of returns in the tails of the distribution—that is where the risk is. In general, greater positive kurtosis and more negative skew in returns distributions indicates increased risk.

## Measures of Sample Skew and Kurtosis

**Sample skewness** is equal to the sum of the cubed deviations from the mean divided by the cubed standard deviation and by the number of observations. Sample skewness for large

samples is computed as:

$$\text{sample skewness} = \frac{1}{n} \frac{\sum_{i=1}^{n} (x_i - \bar{x})^3}{s^3}$$

where:
s = sample standard deviation

Note that the denominator is always positive, but that the numerator can be positive or negative, depending on whether observations above the mean or observations below the mean tend to be further from the mean on average. When a distribution is right skewed, sample skewness is positive because the deviations above the mean are larger on average. A left-skewed distribution has a negative sample skewness.

Dividing by standard deviation cubed standardizes the statistic and allows **interpretation of the skewness measure**. If relative skewness is equal to zero, the data is not skewed. Positive levels of relative skewness imply a positively skewed distribution, whereas negative values of relative skewness imply a negatively skewed distribution. Values of sample skewness in excess of 0.5 in absolute value are considered significant.

**Sample kurtosis** is measured using deviations raised to the *fourth power*.

$$\text{sample kurtosis} = \frac{1}{n} \frac{\sum_{i=1}^{n} (x_i - \bar{x})^4}{s^4}$$

where:
s = sample standard deviation

To **interpret kurtosis**, note that it is measured relative to the kurtosis of a normal distribution, which is 3. Positive values of excess kurtosis indicate a distribution that is leptokurtic (more peaked, fat tails), whereas negative values indicate a platykurtic distribution (less peaked, thin tails). Excess kurtosis values that exceed 1.0 in absolute value are considered large. We can calculate kurtosis relative to that of a normal distribution as:

excess kurtosis = sample kurtosis − 3

### LOS 7.m: Compare the use of arithmetic and geometric means when analyzing investment returns.

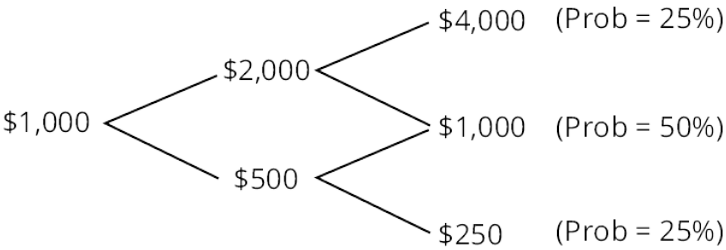
*CFA® Program Curriculum, Volume 1, page 434*

Since past annual returns are compounded each period, the geometric mean of past annual returns is the appropriate measure of past performance. It gives us the average annual *compound* return. With annual returns of 10%, 15%, –4%, 9%, and 5% over five years, the geometric mean return of $[(1.10)(1.15)(0.96)(1.09)(1.05)]^{1/5} - 1 = 6.81\%$ tells us the single rate that, if compounded over the five periods, would lead to the same increase in wealth as the individual annual rates of return.

The arithmetic mean of (10% + 15% – 4% + 9% + 5%) / 5 = 7% is, however, the statistically best estimator of the *next* year's returns given only the five years of return outcomes. To estimate *multi-year* returns (e.g. expected annual return over the next three years), the geometric mean of 6.81% is the appropriate measure.

Consider also a forward-looking model where returns will be either +100% or –50% each year for two years. Note that when returns are +100% in Year 1 and –50% in Year 2, the annual geometric mean return is $[(1 + 100\%)(1 - 50\%)]^{1/2} - 1 = 0\%$ and the arithmetic mean annual return is $(100\% - 50\%)/2 = 25\%$.

Consider the following tree model for two years where the outcomes of +100% and –50% are equally likely and beginning wealth is $1,000:




Each of the four possible ending wealth values has a 25% probability, so the expected ending wealth is simply (4,000 + 1,000 + 1,000 + 250)/4 = $1,562.50, or 0.25(4,000) + 0.50(1,000) + 0.25(250) = $1,562.50. This is consistent with earning a compound annual rate of return equal to the arithmetic mean of 25%, $1.25^2 \times \$1{,}000 = \$1{,}562.50$, but not with earning the geometric mean return of 0%. For each single year, the expected rate of return is simply the average of +100% and –50% since those outcomes are equally likely.

## MODULE QUIZ 7.3

*To best evaluate your performance, enter your quiz answers online.*

1. For a skewed distribution, what is the minimum percentage of the observations that will lie between ±2.5 standard deviations of the mean based on Chebyshev's Inequality?
   A. 56%.
   B. 75%.
   C. 84%.
2. A distribution of returns that has a greater percentage of small deviations from the mean and a greater percentage of extremely large deviations from the mean compared to a normal distribution:
   A. is positively skewed.
   B. has positive excess kurtosis.
   C. has negative excess kurtosis.
3. Which of the following is *most accurate* regarding a distribution of returns that has a mean greater than its median?
   A. It is positively skewed.
   B. It is a symmetric distribution.
   C. It has positive excess kurtosis.
4. A normal distribution has kurtosis of:
   A. zero.
   B. one.
   C. three.
5. The most appropriate measure of central tendency for forecasting an investment return in a single future period is:
   A. the harmonic mean.
   B. the arithmetic mean.
   C. the geometric mean.

## KEY CONCEPTS

### LOS 7.a

Descriptive statistics summarize the characteristics of a data set; inferential statistics are used to make probabilistic statements about a population based on a sample.

A population includes all members of a specified group, while a sample is a subset of the population used to draw inferences about the population.

Data may be measured using different scales.

- Nominal scale—data is put into categories that have no particular order.
- Ordinal scale—data is put into categories that can be ordered with respect to some characteristic.
- Interval scale—differences in data values are meaningful, but ratios, such as twice as much or twice as large, are not meaningful.
- Ratio scale—ratios of values, such as twice as much or half as large, are meaningful, and zero represents the complete absence of the characteristic being measured.

### LOS 7.b

Any measurable characteristic of a population is called a parameter.

A characteristic of a sample is given by a sample statistic.

A frequency distribution groups observations into classes, or intervals. An interval is a range of values.

### LOS 7.c

Relative frequency is the percentage of total observations falling within an interval.

Cumulative relative frequency for an interval is the sum of the relative frequencies for all values less than or equal to that interval's maximum value.

### LOS 7.d

A histogram is a bar chart of data that has been grouped into a frequency distribution. A frequency polygon plots the midpoint of each interval on the horizontal axis and the absolute frequency for that interval on the vertical axis, and connects the midpoints with straight lines. The advantage of histograms and frequency polygons is that we can quickly see where most of the observations lie.

### LOS 7.e

The arithmetic mean is the average. $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$. Population mean and sample mean are examples of arithmetic means.

The geometric mean is used to find a compound growth rate. $G = \sqrt[n]{X_1 \times X_2 \times \ldots \times X_n}$ .

The weighted mean weights each value according to its influence. $\overline{X}_W = \sum_{i=1}^{n} w_i X_i$.

The harmonic mean can be used to find an average purchase price, such as dollars per share for equal periodic investments.

$$\overline{X}_H = \frac{N}{\sum_{i=1}^{N} \frac{1}{x_i}}$$

The median is the midpoint of a data set when the data is arranged from largest to smallest.

The mode of a data set is the value that occurs most frequently.

### LOS 7.f

Quantile is the general term for a value at or below which a stated proportion of the data in a distribution lies. Examples of quantiles include:

- Quartiles—the distribution is divided into quarters.
- Quintile—the distribution is divided into fifths.
- Decile—the distribution is divided into tenths.
- Percentile—the distribution is divided into hundredths (percents).

### LOS 7.g

The range is the difference between the largest and smallest values in a data set.

Mean absolute deviation (MAD) is the average of the absolute values of the deviations from the arithmetic mean:

$$\text{MAD} = \frac{\sum_{i=1}^{n} |X_i - \overline{X}|}{n}$$

Variance is defined as the mean of the squared deviations from the arithmetic mean or from the expected value of a distribution.

- Population variance = $\sigma^2 = \frac{\sum_{i=1}^{N} (X_i - \mu)^2}{N}$, where μ = population mean and N = size.
- Sample variance = $s^2 = \frac{\sum_{i=1}^{N} (X_i - \overline{X})^2}{n-1}$, where $\overline{X}$ = sample mean and n = sample size.

Standard deviation is the positive square root of the variance and is frequently used as a quantitative measure of risk.

### LOS 7.h

Chebyshev's inequality states that the proportion of the observations within *k* standard deviations of the mean is at least $1 - 1/k^2$ for all $k > 1$. It states that for any distribution, at least:

- 36% of observations lie within +/− 1.25 standard deviations of the mean.
- 56% of observations lie within +/− 1.5 standard deviations of the mean.
- 75% of observations lie within +/− 2 standard deviations of the mean.
- 89% of observations lie within +/− 3 standard deviations of the mean.
- 94% of observations lie within +/− 4 standard deviations of the mean.

### LOS 7.i

The coefficient of variation for sample data, $CV = \frac{s}{\bar{X}}$ , is the ratio of the standard deviation of the sample to its mean (expected value of the underlying distribution).

### LOS 7.j

Skewness describes the degree to which a distribution is not symmetric about its mean. A right-skewed distribution has positive skewness. A left-skewed distribution has negative skewness.

Sample skew with an absolute value greater than 0.5 is considered significantly different from zero.

### LOS 7.k

For a positively skewed, unimodal distribution, the mean is greater than the median, which is greater than the mode.

For a negatively skewed, unimodal distribution, the mean is less than the median, which is less than the mode.

### LOS 7.l

Kurtosis measures the peakedness of a distribution and the probability of extreme outcomes (thickness of tails).

- Excess kurtosis is measured relative to a normal distribution, which has a kurtosis of 3.
- Positive values of excess kurtosis indicate a distribution that is leptokurtic (fat tails, more peaked) so that the probability of extreme outcomes is greater than for a normal distribution.
- Negative values of excess kurtosis indicate a platykurtic distribution (thin tails, less peaked).
- Excess kurtosis with an absolute value greater than 1 is considered significant.

### LOS 7.m

The arithmetic mean return is appropriate for forecasting single period returns in future periods, while the geometric mean is appropriate for forecasting future compound returns over multiple periods.

# ANSWER KEY FOR MODULE QUIZZES

## Module Quiz 7.1

1. **B** Poker hand rankings are an ordinal scale. We can say a pair outranks a high card (therefore, it is not a nominal scale), but we cannot say a pair is some amount greater than a high card (therefore, it is not an interval scale). (LOS 7.a)

2. **C** Intervals within a frequency distribution should always be nonoverlapping and closed ended so that each data value can be placed into only one interval. Intervals have no set width and should be set at a width so that data is adequately summarized without losing valuable characteristics. (LOS 7.b)

3. **B** An interval is the set of return values that an observation falls within. Simply count the return intervals on the table—there are five of them. (LOS 7.c)

4. **C** The sample size is the sum of all of the frequencies in the distribution, or 3 + 7 + 3 + 2 + 1 = 16. (LOS 7.c)

5. **C** The relative frequency is found by dividing the frequency of the interval by the total number of frequencies. $\frac{7}{16} = 43.8\%$ (LOS 7.c)

6. **A** In a histogram, the intervals are on the horizontal axis and the frequency is on the vertical axis. (LOS 7.d)

## Module Quiz 7.2

1. **A** [22% + 5% + –7% + 11% + 2% +11%] / 6 = 7.3% (LOS 7.e)

2. **B** To find the median, rank the returns in order and take the middle value: –7%, 2%, 5%, 11%, 11%, 22%. In this case, because there is an even number of observations, the median is the average of the two middle values, or (5% + 11%) / 2 = 8.0%. (LOS 7.e)

3. **C** The mode is the value that appears most often, or 11%. (LOS 7.e)

4. **C** The range is calculated by taking the highest value minus the lowest value.

   22% − (–7%) = 29.0% (LOS 7.g)

5. **B** The mean absolute deviation is found by taking the mean of the absolute values of the deviations from the mean.

   (|22 − 7.3| + |5 − 7.3| + |–7 − 7.3| + |11 − 7.3| + |2 − 7.3| + |11 − 7.3|) / 6 = 7.33%

   (LOS 7.g)

6. **C** The population variance, $\sigma^2$, is found by taking the mean of all squared deviations from the mean.

   $\sigma^2 = [(22 - 7.3)^2 + (5 - 7.3)^2 + (-7 - 7.3)^2 + (11 - 7.3)^2 + (2 - 7.3)^2 + (11 - 7.3)^2] / 6 = 80.2\%^2$

   (LOS 7.g)

7. **B** The population standard deviation, σ, is found by taking the square root of the population variance.

$$\sigma = \sqrt{\frac{(22-7.3)^2+(5-7.3)^2+(-7-7.3)^2+(11-7.3)^2+(2-7.3)^2+(11-7.3)^2}{6}}$$

$= (80.2\%^2)^{0.5} = 8.96\%$

(LOS 7.g)

8. **C** The sample variance, $s^2$, uses n – 1 in the denominator.

$s^2 = [(22 - 7.3)^2 + (5 - 7.3)^2 + (-7 - 7.3)^2 + (11 - 7.3)^2 + (2 - 7.3)^2 + (11 - 7.3)^2] / (6 - 1) = 96.3\%^2$

(LOS 7.g)

9. **A** The sample standard deviation, *s*, is the square root of the sample variance.

$$\sigma = \sqrt{\frac{(22-7.3)^2+(5-7.3)^2+(-7-7.3)^2+(11-7.3)^2+(2-7.3)^2+(11-7.3)^2}{6-1}}$$

$= (96.3)^{0.5} = 9.8\%$

(LOS 7.g)

10. **A** (15% + 19% – 8% + 14%) / 4 = 10% (LOS 7.e)

11. **A** (1.15 × 1.19 × 0.92 × 1.14)0.25 − 1 = 9.45% (LOS 7.e)

**PROFESSOR'S NOTE**

This question could have been answered very quickly since the geometric mean must be less than the arithmetic mean computed in the preceding problem.

12. **B** $\overline{X}_H = \frac{3}{\frac{1}{3}+\frac{1}{4}+\frac{1}{5}} = 3.83$ (LOS 7.e)

13. **C** With eight observations, the location of the 65th percentile is:

(8 + 1) × 65 / 100 = 5.85 observations

The fifth observation is 7 and the sixth observation is 9, so the value at 5.85 observations is 7 + 0.85(9 – 7) = 8.7. (LOS 7.f)

## Module Quiz 7.3

1. **C** Applying Chebyshev's inequality, $1 - [1 / (2.5)^2] = 0.84$, or 84%. (LOS 7.h)

2. **B** A distribution that has a greater percentage of small deviations from the mean and a greater percentage of extremely large deviations from the mean will be leptokurtic and will exhibit excess kurtosis (positive). The distribution will be more peaked and have fatter tails than a normal distribution. (LOS 7.j)

3. **A** A distribution with a mean greater than its median is positively skewed, or skewed to the right. The skew "pulls" the mean. *Note: Kurtosis deals with the overall shape of a distribution and not its skewness*. (LOS 7.k)

4. **C** Normal distributions have kurtosis of three and excess kurtosis of zero. (LOS 7.l)

5. **B** The arithmetic mean is appropriate for forecasting single period returns. The geometric mean is appropriate for forecasting compound returns over more than a single period. (LOS 7.m)

The following is a review of the Quantitative Methods (1) principles designed to address the learning outcome statements set forth by CFA Institute. Cross-Reference to CFA Institute Assigned Reading #8.

# READING 8: PROBABILITY CONCEPTS

**Study Session 2**

## EXAM FOCUS

This topic review covers important terms and concepts associated with probability theory. Random variables, events, outcomes, conditional probability, and joint probability are described. Probability rules such as the addition rule and multiplication rule are introduced. These rules are frequently used by finance practitioners. Expected value, standard deviation, covariance, and correlation for individual asset and portfolio returns are discussed. A well-prepared candidate will be able to calculate and interpret these widely used measures. This review also discusses counting rules, which lay the foundation for the binomial probability distribution that is covered in the next topic review.

## MODULE 8.1: CONDITIONAL AND JOINT PROBABILITIES

Video covering this content is available online.

### LOS 8.a: Define a random variable, an outcome, an event, mutually exclusive events, and exhaustive events.

*CFA® Program Curriculum, Volume 1, page 458*

- A **random variable** is an uncertain quantity/number.
- An **outcome** is an observed value of a random variable.
- An **event** is a single outcome or a set of outcomes.
- **Mutually exclusive events** are events that cannot both happen at the same time.
- **Exhaustive events** are those that include all possible outcomes.

Consider rolling a 6-sided die. The number that comes up is a *random variable*. If you roll a 4, that is an *outcome*. Rolling a 4 is an *event*, and rolling an even number is an *event*. Rolling a 4 and rolling a 6 are *mutually exclusive events*. Rolling an even number and rolling an odd number is a set of mutually exclusive and *exhaustive events*.

### LOS 8.b: State the two defining properties of probability and distinguish among empirical, subjective, and a priori probabilities.

*CFA® Program Curriculum, Volume 1, page 459*

There are **two defining properties of probability**.

- The probability of occurrence of any event ($E_i$) is between 0 and 1 (i.e., $0 \le P(E_i) \le 1$).
- If a set of events, $E_1$, $E_2$, … $E_n$, is mutually exclusive and exhaustive, the probabilities of those events sum to 1 (i.e., $\Sigma P(E_i) = 1$).

The first of the defining properties introduces the term $P(E_i)$, which is shorthand for the "probability of event *i*." If $P(E_i) = 0$, the event will never happen. If $P(E_i) = 1$, the event is certain to occur, and the outcome is not random.

The probability of rolling any one of the numbers 1–6 with a fair die is 1/6 = 0.1667 = 16.7%. The set of events—rolling a number equal to 1, 2, 3, 4, 5, or 6—is exhaustive, and the individual events are mutually exclusive, so the probability of this set of events is equal to 1. We are certain that one of the values in this set of events will occur.

An **empirical probability** is established by analyzing past data. An **a priori probability** is determined using a formal reasoning and inspection process. A **subjective probability** is the least formal method of developing probabilities and involves the use of personal judgment. An analyst may know many things about a firm's performance and have expectations about the overall market that are all used to arrive at a subjective probability, such as, "I believe there is a 70% probability that Acme Foods will outperform the market this year." Empirical and a priori probabilities, by contrast, are **objective probabilities**.

### LOS 8.c: State the probability of an event in terms of odds for and against the event.

*CFA® Program Curriculum, Volume 1, page 460*

Stating the **odds** that an event will or will not occur is an alternative way of expressing probabilities. Consider an event that has a probability of occurrence of 0.125, which is one-eighth. The *odds* that the event will occur are $\frac{0.125}{(1-0.125)} = \frac{1/8}{7/8} = \frac{1}{7}$, which we state as, "the odds for the event occurring are one-to-seven." The *odds against* the event occurring are the reciprocal of 1/7, which is seven-to-one.

We can also get the probability of an event from the odds by reversing these calculations. If we know that the odds for an event are one-to-six, we can compute the probability of occurrence as $\frac{1}{1+6} = \frac{1}{7} = 0.1429 = 14.29\%$. Alternatively, the probability that the event will not occur is $\frac{6}{1+6} = \frac{6}{7} = 0.8571 = 85.71\%$.

**PROFESSOR'S NOTE**

While I am quite familiar with the use of odds rather than probabilities at the horse track, I can't remember encountering odds for a stock or bond. The use of odds at the horse track lets you know how much you will win per $1 bet on a horse (less the track's percentage). If you bet on a 15-1 long shot and the horse wins, you will receive $15 and your $1 bet will be returned, so the profit is $15. Of course, if the horse loses, you would lose the $1 you bet and the "profit" is –$1.

One last point is that the expected return on the bet is zero, based on the probability of winning expressed in the odds. The probability of the horse winning when the odds are 15-to-1 is $\frac{1}{15+1} = \frac{1}{16}$ and the probability of the horse losing is 15/16. The expected profit is $\frac{1}{16} \times \$15 + \frac{15}{16} \times \left(-\$1\right) = 0$.

### LOS 8.d: Distinguish between unconditional and conditional probabilities.

*CFA® Program Curriculum, Volume 1, page 462*

- **Unconditional probability** (a.k.a. *marginal probability*) refers to the probability of an event regardless of the past or future occurrence of other events. If we are concerned with the probability of an economic recession, regardless of the occurrence of changes

in interest rates or inflation, we are concerned with the unconditional probability of a recession.

- A **conditional probability** is one where the occurrence of one event affects the probability of the occurrence of another event. For example, we might be concerned with the probability of a recession *given* that the monetary authority increases interest rates. This is a conditional probability. The key word to watch for here is "given." Using probability notation, "the probability of A *given* the occurrence of B" is expressed as P(A | B), where the vertical bar (|) indicates "given," or "conditional upon." For our interest rate example above, the probability of a recession *given* an increase in interest rates is expressed as P(*recession* | *increase in interest rates*). A conditional probability of an occurrence is also called its **likelihood**.

### LOS 8.e: Explain the multiplication, addition, and total probability rules.

*CFA® Program Curriculum, Volume 1, page 463*

The **multiplication rule of probability** is used to determine the joint probability of two events:

$$P(AB) = P(A \mid B) \times P(B)$$

The **addition rule of probability** is used to determine the probability that at least one of two events will occur:

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

The **total probability rule** is used to determine the unconditional probability of an event, given conditional probabilities:

$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \ldots + P(A \mid B_N)P(B_N)$$

where $B_1, B_2, \ldots B_N$ is a mutually exclusive and exhaustive set of outcomes.

### LOS 8.f: Calculate and interpret 1) the joint probability of two events, 2) the probability that at least one of two events will occur, given the probability of each and the joint probability of the two events, and 3) a joint probability of any number of independent events.

*CFA® Program Curriculum, Volume 1, page 463*

The **joint probability** of two events is the probability that they will both occur. We can calculate this from the conditional probability that A will occur given B occurs (a conditional probability) and the probability that B will occur (the unconditional probability of B). This calculation is sometimes referred to as the *multiplication rule of probability*. Using the notation for conditional and unconditional probabilities we can express this rule as:

$$P(AB) = P(A \mid B) \times P(B)$$

This expression is read as follows: "The joint probability of A and B, P(AB), is equal to the conditional probability of A *given* B, P(A | B), times the unconditional probability of B, P(B)."

This relationship can be rearranged to define the conditional probability of A given B as follows:

$$P(A|B) = \frac{P(AB)}{P(B)}$$

**EXAMPLE: Multiplication rule of probability**

Consider the following information:

- P(I) = 0.4, the probability of the monetary authority increasing interest rates (I) is 40%.
- P(R | I) = 0.7, the probability of a recession (R) given an increase in interest rates is 70%.

What is P(RI), the joint probability of a recession *and* an increase in interest rates?

**Answer:**

Applying the multiplication rule, we get the following result:

P(RI) = P(R | I) × P(I)

P(RI) = 0.7 × 0.4

P(RI) = 0.28

Don't let the cumbersome notation obscure the simple logic of this result. If an interest rate increase will occur 40% of the time and lead to a recession 70% of the time when it occurs, the joint probability of an interest rate increase and a resulting recession is (0.4)(0.7) = (0.28) = 28%.

## Calculating the Probability That at Least One of Two Events Will Occur

The *addition rule* for probabilities is used to determine the probability that at least one of two events will occur. For example, given two events, A and B, the addition rule can be used to determine the probability that either A or B will occur. If the events are *not* mutually exclusive, double counting must be avoided by subtracting the joint probability that *both* A and B will occur from the sum of the unconditional probabilities. This is reflected in the following general expression for the addition rule:

P(A or B) = P(A) + P(B) − P(AB)

For mutually exclusive events, where the joint probability, P(AB), is zero, the probability that either A or B will occur is simply the sum of the unconditional probabilities for each event, P(A or B) = P(A) + P(B).

**Figure 8.1: Venn Diagram for Events That Are Not Mutually Exclusive**

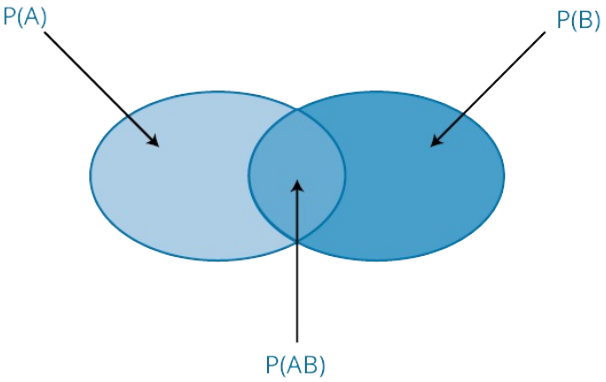



Figure 8.1 illustrates the addition rule with a Venn Diagram and highlights why the joint probability must be subtracted from the sum of the unconditional probabilities. Note that if

the events are *mutually exclusive* the sets do not intersect, P(AB) = 0, and the probability that one of the two events will occur is simply P(A) + P(B).

**EXAMPLE: Addition rule of probability**

Using the information in our previous interest rate and recession example and the fact that the unconditional probability of a recession, P(R), is 34%, determine the probability that either interest rates will increase *or* a recession will occur.

**Answer:**

Given that P(R) = 0.34, P(I) = 0.40, and P(RI) = 0.28, we can compute P(R or I) as follows:

P(R or I) = P(R) + P(I) − P(RI)

P(R or I) = 0.34 + 0.40 − 0.28

P(R or I) = 0.46

## Calculating a Joint Probability of any Number of Independent Events

On the roll of two dice, the joint probability of getting two 4s is calculated as:

P(4 on first die and 4 on second die) = P(4 on first die) × P(4 on second die) = 1/6 × 1/6 = 1/36 = 0.0278

On the flip of two coins, the probability of getting two heads is:

P(heads on first coin and heads on second coin) = 1/2 × 1/2 = 1/4 = 0.25

*Hint:* When dealing with *independent events*, the word *and* indicates multiplication, and the word *or* indicates addition. In probability notation:

P(A **or** B) = P(A) + P(B) − P(AB), and P(A **and** B) = P(A) × P(B)

The multiplication rule we used to calculate the joint probability of two independent events may be applied to any number of independent events, as the following example illustrates.

**EXAMPLE: Joint probability for more than two independent events**

What is the probability of rolling three 4s in one simultaneous toss of three dice?

**Answer:**

Since the probability of rolling a 4 for each die is 1/6, the probability of rolling three 4s is:

P(three 4s on the roll of three dice) = 1/6 × 1/6 × 1/6 = 1/216 = 0.00463

### MODULE QUIZ 8.1

*To best evaluate your performance, enter your quiz answers online.*

1. An event that includes all of the possible outcomes is said to be:
   A. random.
   B. exclusive.
   C. exhaustive.
2. Which of the following values *cannot* be the probability of an event?
   A. 0.00.

B. 1.00.
C. 1.25.

3. The probability that the DJIA will increase tomorrow is 2/3. The probability of an increase in the DJIA stated as odds is:
   A. two-to-one.
   B. one-to-three.
   C. two-to-three.
4. The multiplication rule of probability determines the joint probability of two events as the product of:
   A. two conditional probabilities.
   B. two unconditional probabilities.
   C. a conditional probability and an unconditional probability.
5. If events A and B are mutually exclusive, then:
   A. P(A | B) = P(A).
   B. P(AB) = P(A) × P(B).
   C. P(A or B) = P(A) + P(B).
6. At a charity ball, 800 names are put into a hat. Four of the names are identical. On a random draw, what is the probability that one of these four names will be drawn?
   A. 0.004.
   B. 0.005.
   C. 0.010.
7. Two mutually exclusive events:
   A. always occur together.
   B. cannot occur together.
   C. can sometimes occur together.

## MODULE 8.2: CONDITIONAL EXPECTATIONS, CORRELATION

Video covering this content is available online.

### LOS 8.g: Distinguish between dependent and independent events.

*CFA® Program Curriculum, Volume 1, page 467*

**Independent events** refer to events for which the occurrence of one has no influence on the occurrence of the others. The definition of independent events can be expressed in terms of conditional probabilities. Events A and B are independent if and only if:

P(A | B) = P(A), or equivalently, P(B | A) = P(B)

If this condition is not satisfied, the events are dependent events (i.e., the occurrence of one is dependent on the occurrence of the other).

In our interest rate and recession example, recall that events I and R are not independent; the occurrence of I affects the probability of the occurrence of R. In this example, the independence conditions for I and R are violated because:

P(R) = 0.34, but P(R | I) = 0.7; the probability of a recession is greater when there is an increase in interest rates.

The best examples of independent events are found with the a priori probabilities of dice tosses or coin flips. A die has “no memory.” Therefore, the event of rolling a 4 on the second toss is independent of rolling a 4 on the first toss. This idea may be expressed as:

P(4 on second toss | 4 on first toss) = P(4 on second toss) = 1/6 or 0.167

The idea of independent events also applies to flips of a coin:

P(heads on first coin | heads on second coin) = P(heads on first coin) = 1/2 or 0.50

### LOS 8.h: Calculate and interpret an unconditional probability using the total probability rule.

*CFA® Program Curriculum, Volume 1, page 470*

The *total probability rule* highlights the relationship between unconditional and conditional probabilities of mutually exclusive and exhaustive events. It is used to explain the unconditional probability of an event in terms of probabilities that are conditional upon other events.

In general, the unconditional probability of event R, $P(R) = P(R \mid S_1) \times P(S_1) + P(R \mid S_2) \times P(S_2) + \ldots + P(R \mid S_N) \times P(S_N)$, where the set of events $\{S_1, S_2, \ldots S_N\}$ is mutually exclusive and exhaustive.

**EXAMPLE: An investment application of unconditional probability**

Building upon our ongoing example about interest rates and economic recession, we can assume that a recession can only occur with either of the two events—interest rates increase (I) or interest rates do not increase ($I^C$)—since these events are mutually exclusive and exhaustive. $I^C$ is read "the complement of I," which means "not I." Therefore, the probability of $I^C$ is 1 − P(I). It is logical, therefore, that the sum of the two joint probabilities must be the unconditional probability of a recession. This can be expressed as follows:

$$P(R) = P(RI) + P(RI^C)$$

Applying the multiplication rule, we may restate this expression as:

$$P(R) = P(R \mid I) \times P(I) + P(R \mid I^C) \times P(I^C)$$

Assume that $P(R \mid I) = 0.70$, $P(R \mid I^C)$, the probability of recession if interest rates do not rise, is 10% and that $P(I) = 0.40$ so that $P(I^C) = 0.60$. The unconditional probability of a recession can be calculated as follows:

$$P(R) = P(R \mid I) \times P(I) + P(R \mid I^C) \times P(I^C)$$

$$= (0.70)(0.40) + (0.10)(0.60)$$

$$= 0.28 + 0.06 = 0.34$$

## Expected Value

The expected value of a random variable is the weighted average of the possible outcomes for the variable. The mathematical representation for the expected value of random variable *X* is:

$$E(X) = \Sigma P(x_i)x_i = P(x_1)x_1 + P(x_2)x_2 + \ldots + P(x_n)x_n$$

**EXAMPLE: Expected earnings per share**

The probability distribution of EPS for Ron's Stores is given in the figure below. Calculate the expected earnings per share.

**EPS Probability Distribution**

| Probability | Earnings Per Share |
|---|---|
| 10% | £1.80 |
| 20% | £1.60 |
| 40% | £1.20 |
| 30% | £1.00 |
| 100% | |

**Answer:**

The expected EPS is simply a weighted average of each possible EPS, where the weights are the probabilities of each possible outcome.

E[EPS] = 0.10(1.80) + 0.20(1.60) + 0.40(1.20) + 0.30(1.00) = £1.28

## LOS 8.i: Explain the use of conditional expectation in investment applications.

*CFA® Program Curriculum, Volume 1, page 474*

**Expected values or returns can be calculated using conditional probabilities.** As the name implies, *conditional expected values* are contingent upon the outcome of some other event. An analyst would use a conditional expected value to revise his expectations when new information arrives.

Consider the effect of a tariff on steel imports on the returns of a domestic steel stock. The stock's expected return, given that the government imposes the tariff, will be higher than the expected return if the tariff is not imposed.

Using the total probability rule, we can estimate the (unconditional) expected return on the stock as the sum of the expected return given no tariff times the probability a tariff will not be enacted plus the expected return given a tariff times the probability a tariff will be enacted.

## LOS 8.j: Explain the use of a tree diagram to represent an investment problem.

*CFA® Program Curriculum, Volume 1, page 475*

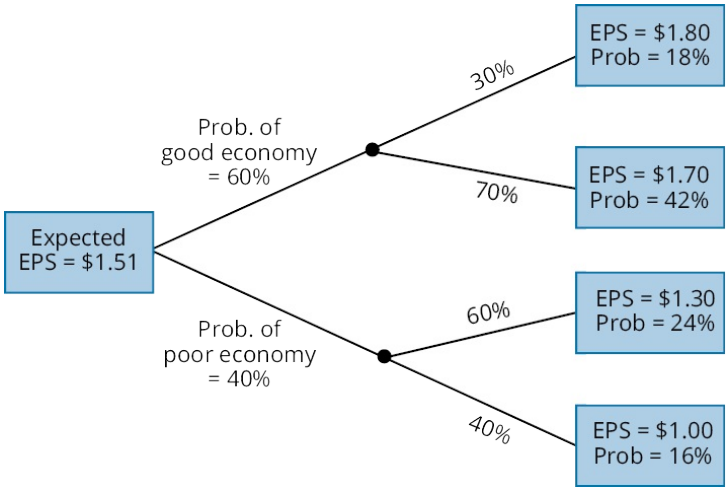
You might well wonder where the returns and probabilities used in calculating expected values come from. A general framework called a **tree diagram** is used to show the probabilities of various outcomes. In Figure 8.2, we have shown estimates of EPS for four different outcomes: (1) a good economy and relatively good results at the company, (2) a good economy and relatively poor results at the company, (3) a poor economy and relatively good results at the company, and (4) a poor economy and relatively poor results at the company. Using the rules of probability, we can calculate the probabilities of each of the four EPS outcomes shown in the boxes on the right-hand side of the "tree."

The expected EPS of $1.51 is simply calculated as:

$$0.18 \times 1.80 + 0.42 \times 1.70 + 0.24 \times 1.30 + 0.16 \times 1.00 = \$1.51$$

Note that the probabilities of the four possible outcomes sum to 1.

**Figure 8.2: A Tree Diagram**




## LOS 8.k: Calculate and interpret covariance and correlation and interpret a scatterplot.

*CFA® Program Curriculum, Volume 1, page 480*

The variance and standard deviation measure the dispersion, or volatility, of only one variable. In many finance situations, however, we are interested in how two random variables move in relation to each other. For investment applications, one of the most frequently analyzed pairs of random variables is the returns of two assets. Investors and managers frequently ask questions such as, "what is the relationship between the return for Stock A and Stock B?" or, "what is the relationship between the performance of the S&P 500 and that of the automotive industry?"

**Covariance** is a measure of how two assets move together. It is the expected value of the product of the deviations of the two random variables from their respective expected values. A common symbol for the covariance between random variables *X* and *Y* is Cov(X,Y). Since we will be mostly concerned with the covariance of asset returns, the following formula has been written in terms of the covariance of the return of asset *i*, $R_i$, and the return of asset *j*, $R_j$:

$$\mathrm{Cov}(R_i,R_j) = E\{[R_i - E(R_i)][R_j - E(R_j)]\}$$

The following are *properties of the covariance*:

- The covariance is a general representation of the same concept as the variance. That is, the variance measures how a random variable moves with itself, and the covariance measures how one random variable moves with another random variable.
- The covariance of $R_A$ with itself is equal to the variance of $R_A$; that is, $\mathrm{Cov}(R_A,R_A) = \mathrm{Var}(R_A)$.
- The covariance may range from negative infinity to positive infinity.

To aid in the interpretation of covariance, consider the returns of a stock and of a put option on the stock. These two returns will have a negative covariance because they move in opposite directions. The returns of two automotive stocks would likely have a positive covariance, and the returns of a stock and a riskless asset would have a zero covariance because the riskless asset's returns never move, regardless of movements in the stock's return. While the formula for covariance given previously is correct, the method of computing the covariance of returns from a joint probability model uses a probability-

weighted average of the products of the random variable's deviations from their means for each possible outcome. The following example illustrates this calculation.

**EXAMPLE: Covariance**

Assume that the economy can be in three possible states (*S*) next year: boom, normal, or slow economic growth. An expert source has calculated that P(boom) = 0.30, P(normal) = 0.50, and P(slow) = 0.20. The returns for Stock A, $R_A$, and Stock B, $R_B$, under each of the economic states are provided in the probability model as follows. What is the covariance of the returns for Stock A and Stock B?

**Probability Distribution of Returns**

| Event | P(S) | $R_A$ | $R_B$ |
|---|---|---|---|
| Boom | 0.3 | 0.20 | 0.30 |
| Normal | 0.5 | 0.12 | 0.10 |
| Slow | 0.2 | 0.05 | 0.00 |

**Answer:**

First, the expected returns for each of the stocks must be determined.

$E(R_A) = (0.3)(0.20) + (0.5)(0.12) + (0.2)(0.05) = 0.13$

$E(R_B) = (0.3)(0.30) + (0.5)(0.10) + (0.2)(0.00) = 0.14$

The covariance can now be computed using the procedure described in the following table.

**Covariance Computation**

| Event | P(S) | $R_A$ | $R_B$ | $P(S) \times [R_A - E(R_A)] \times [R_B - E(R_B)]$ |
|---|---|---|---|---|
| Boom | 0.3 | 0.20 | 0.30 | (0.3)(0.2 − 0.13)(0.3 − 0.14) = 0.00336 |
| Normal | 0.5 | 0.12 | 0.10 | (0.5)(0.12 − 0.13)(0.1 − 0.14) = 0.00020 |
| Slow | 0.2 | 0.05 | 0.00 | (0.2)(0.05 – 0.13)(0 – 0.14) = 0.00224 |
| $Cov(R_A, R_B) = \sum\{P(S) \times [R_A - E(R_A)] \times [R_B - E(R_B)]\} = 0.00580.$ | | | | |

So far, we have calculated covariance from a probability model. We can also determine covariance using historical data. The calculation of the **sample covariance** is based on the following formula:

$$Cov_{1,2} = \frac{\sum_{t=1}^{n}\{[R_{t,1} - \bar{R}_1][R_{t,2} - \bar{R}_2]\}}{n-1}$$

where:

$R_{t,1}$ = return on Asset 1 in period *t*

$R_{t,2}$ = return on Asset 2 in period *t*

$\bar{R}1$ = mean return on Asset 1

$\bar{R}2$ = mean return on Asset 2

n = number of periods

In practice, the covariance is difficult to interpret. This is mostly because it can take on extremely large values, ranging from negative to positive infinity, and, like the variance, these values are expressed in terms of square units.

To make the covariance of two random variables easier to interpret, it may be divided by the product of the random variables' standard deviations. The resulting value is called the **correlation coefficient**, or simply, correlation. The relationship between covariances, standard deviations, and correlations can be seen in the following expression for the correlation of the returns for asset *i* and *j*:

$$\text{Corr}\left(R_i, R_j\right) = \frac{\text{Cov}\left(R_i, R_j\right)}{\sigma(R_i)\sigma(R_j)}$$

$$\text{which implies Cov}\left(R_i, R_j\right) = \text{Corr}\left(R_i, R_j\right)\sigma\left(R_i\right)\sigma\left(R_j\right)$$

The correlation between two random return variables may also be expressed as $\rho(R_i,R_j)$, or $\rho_{i,j}$. Correlation can be forward-looking if it uses covariance from a probability model, or backward-looking if it uses sample covariance from historical data.

*Properties of correlation* of two random variables $R_i$ and $R_j$ are summarized here:

- Correlation measures the strength of the linear relationship between two random variables.
- Correlation has no units.
- The correlation ranges from –1 to +1. That is, $-1 \leq \text{Corr}(R_i, R_j) \leq +1$.
- If $\text{Corr}(R_i, R_j) = 1.0$, the random variables have perfect positive correlation. This means that a movement in one random variable results in a proportional positive movement in the other relative to its mean.
- If $\text{Corr}(R_i, R_j) = -1.0$, the random variables have perfect negative correlation. This means that a movement in one random variable results in an exact opposite proportional movement in the other relative to its mean.
- If $\text{Corr}(R_i, R_j) = 0$, there is no linear relationship between the variables, indicating that prediction of $R_i$ cannot be made on the basis of $R_j$ using linear methods.

**EXAMPLE: Correlation**

Using our previous example, compute and interpret the correlation of the returns for stocks A and B, given that $\sigma^2(R_A) = 0.0028$ and $\sigma^2(R_B) = 0.0124$ and recalling that $\text{Cov}(R_A, R_B) = 0.0058$.

**Answer:**

First, it is necessary to convert the variances to standard deviations.

$$\sigma(R_A) = (0.0028)^{1/2} = 0.0529$$

$$\sigma(R_B) = (0.0124)^{1/2} = 0.1114$$

Now, the correlation between the returns of Stock A and Stock B can be computed as follows:

$$\text{Corr}\left(R_A, R_B\right) = \frac{0.0058}{(0.0529)(0.1114)} = 0.9842$$

The fact that this value is close to +1 indicates that the linear relationship is not only positive but very strong.

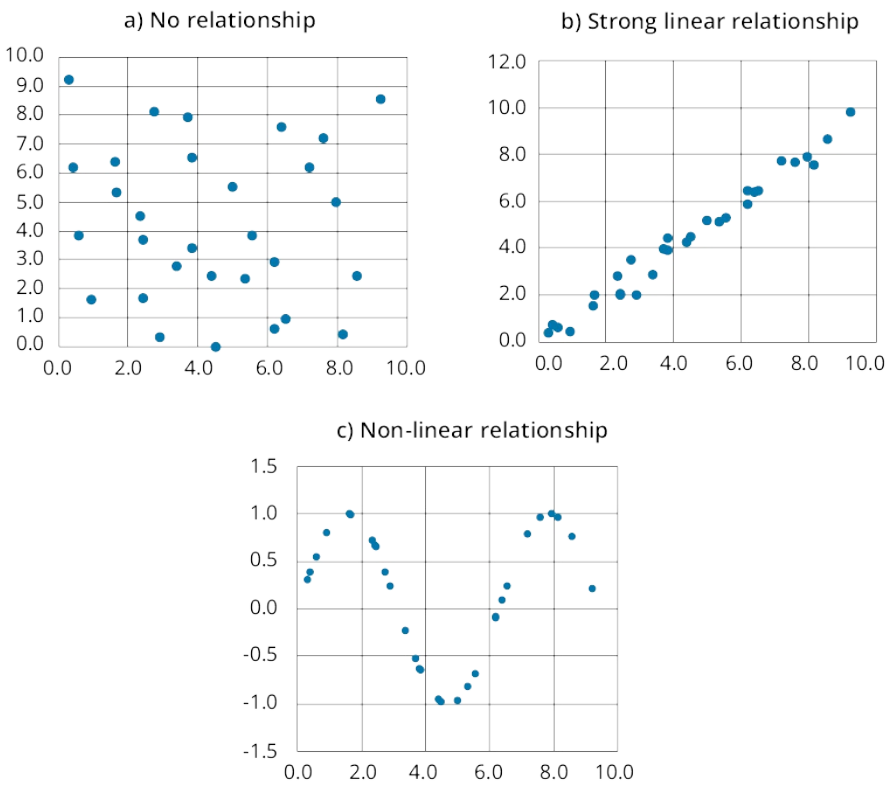
**Scatterplots** are a method for displaying the relationship between two variables. With one variable on the vertical axis and the other on the horizontal axis, their paired observations can

each be plotted as a single point. For example, in panel a of Figure 8.3, the point farthest to the upper right shows that when one of the variables (on the horizontal axis) equaled 9.2, the other variable (on the vertical axis) equaled 8.5.

The scatterplot in panel a is typical of two variables that have no clear relationship. Panel b shows two variables that have a strong linear relationship, that is, a high correlation coefficient.

A key advantage of creating scatterplots is that they can reveal non-linear relationships, which are not described by the correlation coefficient. Panel c illustrates such a relationship. Although the correlation coefficient for these two variables is close to zero, their scatterplot shows clearly that they are related in a predictable way.

**Figure 8.3: Scatterplots**




Care should be taken when drawing conclusions based on correlation. Causation isn't implied just from significant correlation. Even if it were, which variable is "causing" change in the other isn't revealed by correlation. It is more prudent to say that two variables exhibit positive (or negative) association, suggesting that the nature of any causal relationship is to be separately investigated or based on theory that can be subject to additional tests.

One question that can be investigated is the role of outliers (extreme values) in the correlation of two variables. If removing the outliers significantly reduces the calculated correlation, further inquiry is necessary into whether the outliers provide information or are caused by noise (randomness) in the data used.

**Spurious correlation** refers to correlation that is either the result of chance or present due to changes in both variables over time that is caused by their association with a third variable.

For example, we can find instances where two variables that are both related to the inflation rate exhibit significant correlation, but for which causation in either direction is not the case.

In his book *Spurious Correlation*[1], Tyler Vigen presents the following examples. The correlation between the age of each year's Miss America and the number of films Nicholas Cage appeared in that year is 87%. This seems a bit random. The correlation between the U.S. spending on science, space, and technology and suicides by hanging, strangulation, and suffocation over the 1999–2009 period is 99.87%. Impressive correlation, but both variables increased in an approximately linear fashion over the period.

### PROFESSOR'S NOTE

Here, the passage of time seems to be the "third variable" driving the results. Why, I have no idea, but I'm 99.87% sure that neither of the two variables is the cause of the other.

### MODULE QUIZ 8.2

*To best evaluate your performance, enter your quiz answers online.*

1. Two events are said to be independent if the occurrence of one event:
   A. means that the second event cannot occur.
   B. means that the second event is certain to occur.
   C. does not affect the probability of the occurrence of the other event.
2. An analyst estimates that a share price has an 80% probability of increasing if economic growth exceeds 3%, a 40% probability of increasing if economic growth is positive but less than 3%, and a 10% probability of increasing if economic growth is negative. If economic growth has a 25% probability of exceeding 3% and a 25% probability of being negative, what is the probability that the share price increases?
   A. 22.5%.
   B. 42.5%.
   C. 62.5%.
3. Possible values for the covariance of two random variables are:
   A. unbounded.
   B. bounded by 0 and 1.
   C. bounded by –1 and 1.

## MODULE 8.3: PORTFOLIO VARIANCE, BAYES, AND COUNTING PROBLEMS

Video covering this content is available online.

**LOS 8.l: Calculate and interpret the expected value, variance, and standard deviation of a random variable and of returns on a portfolio.**

*CFA® Program Curriculum, Volume 1, page 480*

Recall that the expected value of a random variable is the weighted average of its possible outcomes:

$$E(X) = \Sigma P(x_i)x_i = P(x_1)x_1 + P(x_2)x_2 + \ldots + P(x_n)x_n$$

The **expected value and variance for a portfolio of assets** can be determined using the properties of the individual assets in the portfolio. To do this, it is necessary to establish the portfolio weight for each asset. As indicated in the formula, the weight, *w*, of portfolio asset *i* is simply the market value currently invested in the asset divided by the current market value of the entire portfolio.

$$w_i = \frac{\text{market value of investment in asset i}}{\text{market value of the portfolio}}$$

**Portfolio expected value.** The expected value of a portfolio composed of *n* assets with weights, $w_i$, and expected values, $R_i$, can be determined using the following formula:

$$E(R_p) = \sum_{i=1}^{N} w_i E(R_i) = w_1E(R_1) + w_2E(R_2) + \ldots + w_nE(R_n)$$

More often, we have expected returns (rather than expected prices). When the $R_i$ are returns, the expected return for a portfolio, $E(R_P)$, is calculated using the asset weights and the same formula as above.

**Portfolio variance.** The variance of the portfolio return uses the portfolio weights also, but in a more complicated way:

$$Var(R_p) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j Cov(R_i,R_j)$$

The variance of a portfolio composed of risky asset A and risky asset B can be expressed as:

$$Var(R_p) = w_Aw_ACov(R_A,R_A) + w_Aw_BCov(R_A,R_B) + w_Bw_ACov(R_B,R_A) + w_Bw_BCov(R_B,R_B)$$

**PROFESSOR'S NOTE**

Earlier in this Study Session we calculated sample variance from historical data. Here, we are calculating variance based on covariance from a probability model.

Now, since $Cov(R_A,R_B) = Cov(R_B,R_A)$, and $Cov(R_A,R_A) = \sigma^2(R_A)$, this expression reduces to the following:

$$Var(R_P) = w_A{}^2\sigma^2(R_A) + w_B{}^2\sigma^2(R_B) + 2w_Aw_BCov(R_A,R_B)$$

Since $Cov(R_A,R_B) = \sigma(R_B)\sigma(R_A)\rho(R_A,R_B)$, another way to present this formula is:

$$Var(R_P) = w_A{}^2\sigma^2(R_A) + w_B{}^2\sigma^2(R_B) + 2w_Aw_B\sigma(R_A)\sigma(R_B)\rho(R_A,R_B)$$

**EXAMPLE: Standard deviation of a two-asset portfolio**

A portfolio is 30% invested in stocks with a standard deviation of returns of 20%, and the remainder is invested in bonds with a standard deviation of returns of 12%. The correlation of bond returns with stock returns is 0.6. Calculate the standard deviation of returns for the portfolio.

**Answer:**

Portfolio standard deviation is:

$$\sqrt{w_{stocks}^2\sigma_{stocks}^2 + w_{bonds}^2\sigma_{bonds}^2 + 2w_{stocks}w_{bonds}\sigma_{stocks}\sigma_{bonds}\rho_{bonds,\,stocks}}$$

$$= \sqrt{(0.3^2)(0.20^2) + (0.7^2)(0.12^2) + 2(0.3)(0.7)(0.20)(0.12)(0.60)}$$

$$= \sqrt{0.0167} = 12.92\%$$

If the question had provided the covariance of asset returns (0.0144) rather than their correlation, the calculation of portfolio return standard deviation would have been:

$$\sqrt{w_{stocks}^2\sigma_{stocks}^2 + w_{bonds}^2\sigma_{bonds}^2 + 2w_{stocks}w_{bonds}Cov_{bonds,\ stocks}} = \sqrt{(0.3^2)(0.20^2) + (0.7^2)(0}$$

## LOS 8.m: Calculate and interpret covariance given a joint probability function.

*CFA® Program Curriculum, Volume 1, page 489*

**EXAMPLE: Covariance using a joint probability model**

The joint probabilities of the returns of Asset A and Asset B are given in the following figure. Calculate the covariance of returns for Asset A and Asset B.

**Probability Table**

| Joint Probabilities | $R_B = 0.40$ | $R_B = 0.20$ | $R_B = 0.00$ |
|---|---|---|---|
| $R_A = 0.20$ | 0.15 | 0 | 0 |
| $R_A = 0.15$ | 0 | 0.60 | 0 |
| $R_A = 0.04$ | 0 | 0 | 0.25 |

**Answer:**

The expected returns for the individual assets are determined as:

$E(R_A) = P(R_{A1},R_{B1})R_{A1} + P(R_{A2},R_{B2})R_{A2} + P(R_{A3},R_{B3})R_{A3}$

$E(R_A) = (0.15)(0.20) + (0.60)(0.15) + (0.25)(0.04) = 0.13$

$E(R_B) = P(R_{B1},R_{A1})R_{B1} + P(R_{B2},R_{A2})R_{B2} + P(R_{B3},R_{A3})R_{B3}$

$E(R_B) = (0.15)(0.40) + (0.60)(0.20) + (0.25)(0.00) = 0.18$

The covariance of the asset returns is determined as:

$Cov(R_A, R_B) = P(R_{A1},R_{B1})[R_{A1} - E(R_A)][(R_{B1} - E(R_B)]$

$+ P(R_{A2},R_{B2})[R_{A2} - E(R_A)][(R_{B2} - E(R_B)]$

$+ P(R_{A3},R_{B3})[R_{A3} - E(R_A)][(R_{B3} - E(R_B)]$

$Cov(R_A, R_B) = 0.15(0.20 - 0.13)(0.40 - 0.18)$

$+ 0.60(0.15 - 0.13)(0.20 - 0.18)$

$+ 0.25(0.04 - 0.13)(0.00 - 0.18)$

$= 0.0066$

## LOS 8.n: Calculate and interpret an updated probability using Bayes' formula.

*CFA® Program Curriculum, Volume 1, page 491*

**Bayes' formula** is used to update a given set of prior probabilities for a given event in response to the arrival of new information. The rule for updating prior probability of an event is:

updated probability =

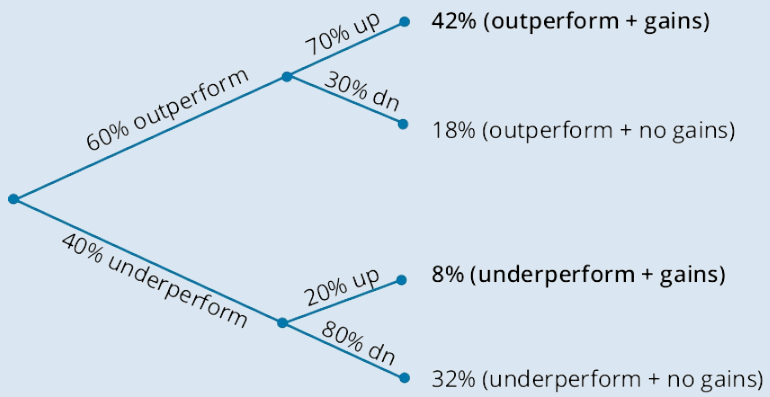
$$\frac{\text{probability of new information for a given event}}{\text{unconditional probability of new information}} \times \text{prior probability of event}$$

Note in the following example of the application of Bayes' formula that we can essentially reverse a given set of conditional probabilities. This means that given P(B), P(A | B), and P(A | $B^C$), it is possible to use Bayes' formula to compute P(B | A).

**EXAMPLE: Bayes' formula**

There is a 60% probability the economy will outperform, and if it does, there is a 70% chance a stock will go up and a 30% chance the stock will go down. There is a 40% chance the economy will underperform, and if it does, there is a 20% chance the stock in question will increase in value (have gains) and an 80% chance it will not. Given that the stock increased in value, calculate the probability that the economy outperformed.

**Answer:**




In the figure above, we have multiplied the probabilities to calculate the probabilities of each of the four outcome pairs. Note that these sum to 1. Given that the stock has gains, what is our updated probability of an outperforming economy? We sum the probability of stock gains in both states (outperform and underperform) to get 42% + 8% = 50%. Given that the stock has gains, the probability that the economy has outperformed is $\frac{42\%}{50\%} = 84\%$.

## LOS 8.o: Identify the most appropriate method to solve a particular counting problem and solve counting problems using factorial, combination, and permutation concepts.

*CFA® Program Curriculum, Volume 1, page 495*

**Labeling** refers to the situation where there are *n* items that can each receive one of *k* different labels. The number of items that receives label 1 is $n_1$ and the number that receive label 2 is $n_2$, and so on, such that $n_1 + n_2 + n_3 + \ldots + n_k = n$. The total number of ways that the labels can be assigned is:

$$\frac{n!}{(n_1!) \times (n_2!) \times \ldots \times (n_k!)}$$

where:

the symbol "!" stands for **factorial**. For example, 4! = 4 × 3 × 2 × 1 = 24, and 2! = 2 × 1 = 2.

The general expression for *n* factorial is:

$n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \ldots \times 1$, where by definition, 0! = 1

*Calculator help:* On the TI, factorial is [2nd] [x!] (above the multiplication sign). On the HP, factorial is [g] [n!]. To compute 4! on the TI, enter [4][2nd][x!] = 24. On the HP, press [4] [ENTER][g][n!].

**EXAMPLE: Labeling**

Consider a portfolio consisting of eight stocks. Your goal is to designate four of the stocks as "long-term holds," three of the stocks as "short-term holds," and one stock as "sell." How many ways can these eight stocks be labeled?

**Answer:**

There are 8! = 40,320 total possible sequences that can be followed to assign the three labels to the eight stocks. However, the order that each stock is assigned a label does not matter. For example, it does not matter which of the stocks labeled "long-term" is the first to be labeled. Thus, there are 4! ways to assign the long-term label. Continuing this reasoning to the other categories, there are 4! × 3! × 1! equivalent sequences for assigning the labels. To eliminate the counting of these redundant sequences, the total number of possible sequences (8!) must be divided by the number of redundant sequences (4! × 3! × 1!).

Thus, the number of *different* ways to label the eight stocks is:

$$\frac{8!}{4!\times 3!\times 1!} = \frac{40{,}320}{24\times 6\times 1} = 280$$

If there are *n* labels (k = n), we have $\frac{n!}{1} = n!$. The number of ways to assign *n* different labels to *n* items is simply *n!*.

A special case of labeling arises when the number of labels equals 2 (k = 2). That is, the *n* items can only be in one of two groups, and $n_1 + n_2 = n$. In this case, we can let $r = n_1$ and $n_2 = n - r$. Since there are only two categories, we usually talk about choosing *r* items. Then (n − r) items are not chosen. The general formula for labeling when k = 2 is called the **combination formula** (or *binomial formula*) and is expressed as:

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

where ${}_nC_r$ is the number of possible ways (combinations) of selecting *r* items from a set of *n* items when the order of selection is not important. This is also written $\binom{n}{r}$ and read "n choose r."

Another useful formula is the **permutation formula**. A permutation is a specific ordering of a group of objects. The question of how many different groups of size *r* in specific order can be chosen from *n* objects is answered by the permutation formula. The number of permutations of *r* objects from *n* objects = $\frac{n!}{(n-r)!}$. We will give an example using this formula shortly.

**PROFESSOR'S NOTE**

The combination formula ${}_nC_r$ and the permutation formula ${}_nP_r$ are both available on the TI calculator. To calculate the number of different groups of three stocks from a list of eight stocks (i.e., ${}_8C_3$), the sequence is 8 [2nd] [${}_nC_r$] 3 [=], which yields 56. If we want to know the number of differently ordered groups of three that can be selected from a list of eight, we enter 8 [2nd] [${}_nP_r$] 3 [=] to get 336, which is the number of permutations, $\frac{8!}{(8-3)!}$. This function is not available on the HP calculator. Remember, current policy permits you to bring both calculators to the exam if you choose.

**EXAMPLE: Number of choices in any order**

How many ways can three stocks be sold from an 8-stock portfolio?

**Answer:**

This is similar to the preceding labeling example. Since order does not matter, we take the total number of possible ways to select three of the eight stocks and divide by the number of possible redundant selections. Thus, the answer is:

$$\frac{8!}{5! \times 3!} = 56$$

In the preceding two examples, ordering did not matter. The order of selection could, however, be important. For example, suppose we want to liquidate only one stock position per week over the next three weeks. Once we choose three particular stocks to sell, the order in which they are sold must be determined. In this case, the concept of permutation comes into play. The *permutation formula* is:

$${}_{n}P_{r} = \frac{n!}{(n-r)!}$$

where ${}_{n}P_{r}$ is the number of possible ways (permutations) to select *r* items from a set of *n* items when the order of selection is important. The permutation formula implies that there are r! *more* ways to choose *r* items if the order of selection *is important* than if order is not important. That is, ${}_{n}P_{r} = r! \times {}_{n}C_{r}$ .

**EXAMPLE: Permutation**

How many ways are there to sell three stocks out of eight if the order of the sales is important?

**Answer:**

$${}_{n}P_{r} = {}_{8}P_{3} = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 336$$

This is 3! times the 56 possible combinations computed in the preceding example for selecting the three stocks when the order was not important.

There are five guidelines that may be used to determine which counting method to employ when dealing with counting problems:

- The *multiplication rule of counting* is used when there are *two or more groups*. The key is that only *one* item may be selected from each group. If there are *k* steps required to complete a task and each step can be done in *n* ways, the number of different ways to complete the task is $n_1 \times n_2 \times \ldots \times n_k$.
- *Factorial* is used by itself when there are *no groups*—we are only arranging a given set of n items. Given n items, there are n! ways of arranging them.
- The *labeling formula* applies to *three or more subgroups* of predetermined size. Each element of the entire group must be assigned a place, or label, in one of the three or more subgroups.
- The *combination formula* applies to *only two groups* of predetermined size. Look for the word “choose” or “combination.”
- The *permutation formula* applies to *only two groups* of predetermined size. Look for a specific reference to “order” being important.

## MODULE QUIZ 8.3

*To best evaluate your performance, enter your quiz answers online.*

1. Given the conditional probabilities in the table below and the unconditional probabilities P(Y = 1) = 0.3 and P(Y = 2) = 0.7, what is the expected value of *X*?

| $x_i$ | $P(x_i \mid Y = 1)$ | $P(x_i \mid Y = 2)$ |
|---|---|---|
| 0 | 0.2 | 0.1 |
| 5 | 0.4 | 0.8 |
| 10 | 0.4 | 0.1 |

   A. 5.0.
   B. 5.3.
   C. 5.7.

2. A discrete uniform distribution (each event has an equal probability of occurrence) has the following possible outcomes for X: [1, 2, 3, 4]. The variance of this distribution is *closest* to:
   A. 1.00.
   B. 1.25.
   C. 2.00.

3. The correlation of returns between Stocks A and B is 0.50. The covariance between these two securities is 0.0043, and the standard deviation of the return of Stock B is 26%. The variance of returns for Stock A is:
   A. 0.0011.
   B. 0.0331.
   C. 0.2656.

4. An analyst believes Davies Company has a 40% probability of earning more than $2 per share. She estimates that the probability that Davies Company's credit rating will be upgraded is 70% if its earnings per share are greater than $2 and 20% if its earnings per share are $2 or less. Given the information that Davies Company's credit rating has been upgraded, what is the updated probability that its earnings per share are greater than $2?
   A. 50%.
   B. 60%.
   C. 70%.

5. Consider a universe of 10 bonds from which an investor will ultimately purchase six bonds for his portfolio. If the order in which he buys these bonds is not important, how many potential 6-bond combinations are there?
   A. 7.
   B. 210.
   C. 5,040.

6. There are 10 sprinters in the finals of a race. How many different ways can the gold, silver, and bronze medals be awarded?
   A. 120.
   B. 720.
   C. 1,440.

## KEY CONCEPTS

### LOS 8.a

A random variable is an uncertain value determined by chance.

An outcome is the realization of a random variable.

An event is a set of one or more outcomes. Two events that cannot both occur are termed "mutually exclusive" and a set of events that includes all possible outcomes is an "exhaustive" set of events.

### LOS 8.b

The two properties of probability are:

- The sum of the probabilities of all possible mutually exclusive events is 1.
- The probability of any event cannot be greater than 1 or less than 0.

A priori probability measures predetermined probabilities based on well-defined inputs; empirical probability measures probability from observations or experiments; and subjective probability is an informed guess.

### LOS 8.c

Probabilities can be stated as odds that an event will or will not occur. If the probability of an event is A out of B trials (A/B), the "odds for" are A to (B − A) and the "odds against" are (B − A) to A.

### LOS 8.d

Unconditional probability (marginal probability) is the probability of an event occurring.

Conditional probability, $P(A \mid B)$, is the probability of an event A occurring given that event B has occurred.

### LOS 8.e

The multiplication rule of probability is used to determine the joint probability of two events:

$$P(AB) = P(A \mid B) \times P(B)$$

The addition rule of probability is used to determine the probability that at least one of two events will occur:

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

The total probability rule is used to determine the unconditional probability of an event, given conditional probabilities:

$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \ldots + P(A \mid B_N)P(B_N)$$

where $B_1, B_2, \ldots B_N$ is a mutually exclusive and exhaustive set of outcomes.

### LOS 8.f

The joint probability of two events, P(AB), is the probability that they will both occur. $P(AB) = P(A \mid B) \times P(B)$. For independent events, $P(A \mid B) = P(A)$, so that $P(AB) = P(A) \times P(B)$.

The probability that at least one of two events will occur is P(A or B) = P(A) + P(B) − P(AB). For mutually exclusive events, P(A or B) = P(A) + P(B), since P(AB) = 0.

The joint probability of any number of independent events is the product of their individual probabilities.

### LOS 8.g

The probability of an independent event is unaffected by the occurrence of other events, but the probability of a dependent event is changed by the occurrence of another event. Events A and B are independent if and only if:

P(A | B) = P(A), or equivalently, P(B | A) = P(B)

### LOS 8.h

Using the total probability rule, the unconditional probability of A is the probability-weighted sum of the conditional probabilities:

$$P(A) = \sum_{i=1}^{n} [P_i(B_i)] \times P(A|B_i)$$

where $B_i$ is a set of mutually exclusive and exhaustive events.

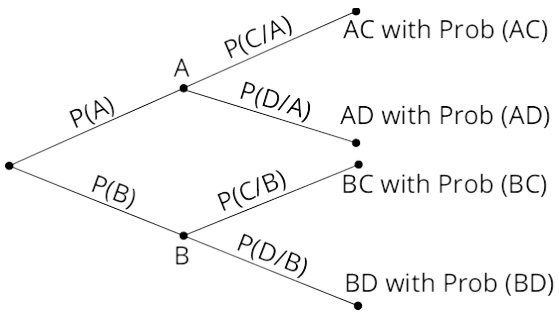
### LOS 8.i

Conditional expected values depend on the outcome of some other event.

Forecasts of expected values for a stock's return, earnings, and dividends can be refined, using conditional expected values, when new information arrives that affects the expected outcome.

### LOS 8.j

A tree diagram shows the probabilities of two events and the conditional probabilities of two subsequent events.




### LOS 8.k

Covariance measures the extent to which two random variables tend to be above and below their respective means for each joint realization. It can be calculated from a probability model as:

$$Cov_{A,B} = \sum_{i=1}^{N} P_i (A_i - \overline{A})(B_i - \overline{B})$$

Sample covariance can be calculated from historical data as:

$$\mathrm{Cov}_{A,B} = \frac{\sum_{i=1}^{n} \{ |A_i - \bar{A}| \, |B_i - \bar{B}| \}}{n-1}$$

Correlation is a standardized measure of association between two random variables. It ranges in value from –1 to +1 and is equal to $\frac{\mathrm{Cov}A,B}{\sigma_A \sigma_B}$.

Scatterplots display the relationship between two variables graphically. They are useful for revealing nonlinear relationships that are not measured by correlation.

Correlation does not imply that changes in one variable cause changes in the other. Spurious correlation may result by chance or from the relationships of two variables to a third variable.

## LOS 8.l

The expected value of a random variable, E(X), equals $\sum_{i=1}^{n} P_i(x_i) X_i$.

The variance of a random variable, Var(X), equals $\sum_{i=1}^{n} P(x_i)[X_i - E(X)]^2 = \sigma_X^2$.

Standard deviation: $\sigma_X = \sqrt{\sigma_X^2}$.

The expected returns and variance of a 2-asset portfolio are given by:

$$\begin{aligned} E(R_P) &= w_1 E(R_1) + w_2 E(R_2) \\ \mathrm{Var}(R_P) &= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \mathrm{Cov}_{1,2} \\ &= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_1 \sigma_2 \rho_{1,2} \end{aligned}$$

## LOS 8.m

Given the joint probabilities for $X_i$ and $Y_i$, i.e., $P(X_i Y_i)$, the covariance is calculated as:

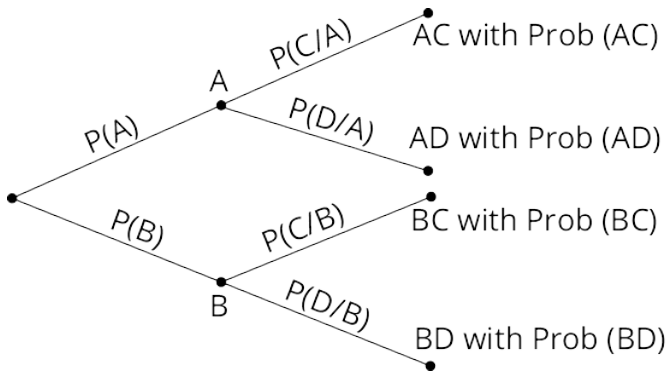
$$\sum_{i=1}^{n} P(X_i Y_i) [X_i - E(X)] [Y_i - E(Y)]$$

## LOS 8.n

Bayes' formula for updating probabilities based on the occurrence of an event *O* is:

$$P(I|O) = \frac{P(O|I)}{P(O)} \times P\left(I\right)$$

Equivalently, based on the tree diagram below, $P(A|C) = \frac{P(AC)}{P(AC)+P(BC)}$.




## LOS 8.o

The number of ways to order *n* objects is *n* factorial, n! = n × (n –1) × (n − 2) × … × 1.

There are $\frac{N!}{n_1! \times n_2! \times \ldots \times n_k!}$ ways to assign *k* different labels to *n* items, where $n_i$ is the number of items with the label *i*.

The number of ways to choose a subset of size *r* from a set of size *n* when order doesn't matter is $\frac{n!}{(n-r)!r!}$ combinations; when order matters, there are $\frac{n!}{(n-r)!}$ permutations.

# ANSWER KEY FOR MODULE QUIZZES

## Module Quiz 8.1

1. **C** An event is said to be exhaustive if it includes all possible outcomes. (LOS 8.a)

2. **C** Probabilities may range from 0 (meaning no chance of occurrence) through 1 (which means a sure thing). (LOS 8.b)

3. **A** Odds for $\mathrm{E} = \mathrm{P}\left(\mathrm{E}\right) / \left[1 - \mathrm{P}\left(\mathrm{E}\right)\right] = \frac{2/3}{1/3} = 2/1 =$ two-to-one (LOS 8.c)

4. **C** By the multiplication rule of probability, the joint probability of two events, P(AB), is the product of a conditional probability, P(A | B), and an unconditional probability, P(B). (LOS 8.d, LOS 8.e)

5. **C** There is no intersection of events when events are mutually exclusive. P(A | B) = P(A) × P(B) is only true for independent events. Note that since A and B are mutually exclusive (cannot both happen), P(A | B) and P(AB) must both be equal to zero. (LOS 8.f)

6. **B** P(name 1 or name 2 or name 3 or name 4) = 1/800 + 1/800 + 1/800 + 1/800 = 4/800 = 0.005. (LOS 8.f)

7. **B** One or the other may occur, but not both. (LOS 8.f)

## Module Quiz 8.2

1. **C** Two events are said to be independent if the occurrence of one event does not affect the probability of the occurrence of the other event. (LOS 8.g)

2. **B** The three outcomes given for economic growth are mutually exclusive and exhaustive. The probability that economic growth is positive but less than 3% is 100% – 25% – 25% = 50%. Using the total probability rule, the probability that the share price increases is (80%)(25%) + (40%)(50%) + (10%)(25%) = 42.5%. (LOS 8.h, LOS 8.i)

3. **A** Covariance may have any value. Correlation is bounded by –1 and 1. (LOS 8.k)

## Module Quiz 8.3

1. **B** E(X | Y = 1) = (0.2)(0) + (0.4)(5) + (0.4)(10) = 6

   E(X | Y = 2) = (0.1)(0) + (0.8)(5) + (0.1)(10) = 5

   E(X) = (0.3)(6) + (0.7)(5) = 5.30

   (LOS 8.l)

2. **B** Expected value = (1/4)(1 + 2 + 3 + 4) = 2.5

   Variance = (1/4)[(1 − 2.5)2 + (2 − 2.5)2 + (3 − 2.5)2 + (4 − 2.5)2] = 1.25

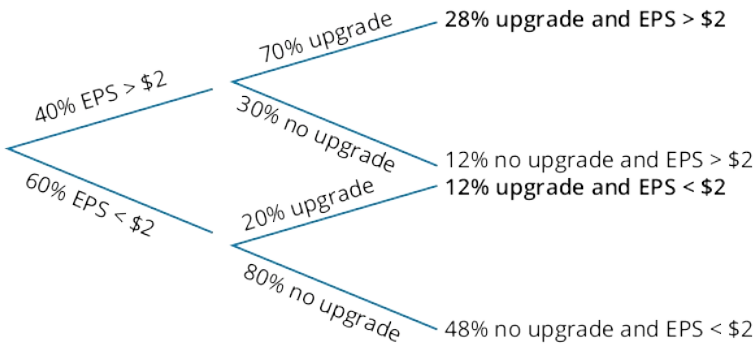
   Note that since each observation is equally likely, each has 25% (1/4) chance of occurrence. (LOS 8.l)

3. **A** $\text{Corr}(R_A,R_B) = \frac{\text{Cov}(R_A,R_B)}{[\sigma(R_A)][\sigma(R_B)]}$

$\sigma^2(R_A) = \left[\frac{\text{Cov}(R_A,R_B)}{\sigma(R_B)\,\text{Corr}(RA,R_B)}\right]^2 = \left[\frac{0.0043}{(0.26)(0.5)}\right]^2 = 0.0331^2 = 0.0011$

(LOS 8.l)

4. **C** This is an application of Bayes' formula. As the tree diagram below shows, the updated probability that earnings per share are greater than \$2 is $\frac{28\%}{28\%+12\%} = 70\%$.




(LOS 8.n)

6. **B** ${}_nC_r = \frac{n!}{(n-r)!r!} = {}_{10}C_6 = \frac{10!}{(10-6)!6!} = \frac{10!}{4!6!} = 210$ (LOS 8.o)

7. **B** Since the order of the top three finishers matters, we need to use the permutation formula.

$10P_3 = \frac{10!}{(10-3)!} = 720$

(LOS 8.o)

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1. "Spurious Correlations," Tyler Vigen, www.tylervigen.com

The following is a review of the Quantitative Methods (2) principles designed to address the learning outcome statements set forth by CFA Institute. Cross-Reference to CFA Institute Assigned Reading #9.

# READING 9: COMMON PROBABILITY DISTRIBUTIONS

**Study Session 3**

## EXAM FOCUS

This topic review contains a lot of key material. Learn the difference between discrete and continuous probability distributions. The binomial and normal distributions are the most important here. You must learn the properties of both distributions and memorize the formulas for the probability of a particular value when given a binomial probability distribution. Learn what shortfall risk is and how to calculate and use Roy's safety-first criterion. Know how to standardize a normally distributed random variable, use a *z*-table, and construct confidence intervals. These skills will be used repeatedly in the topic reviews that follow. Additionally, understand the basic features of the lognormal distribution, Monte Carlo simulation, and historical simulation. Finally, it would be a good idea to know how to get continuously compounded rates of return from holding period returns. Other than that, no problem.

## MODULE 9.1: UNIFORM AND BINOMIAL DISTRIBUTIONS

Video covering this content is available online.

**LOS 9.a: Define a probability distribution and distinguish between discrete and continuous random variables and their probability functions.**

**LOS 9.b: Describe the set of possible outcomes of a specified discrete random variable.**

*CFA® Program Curriculum, Volume 1, page 514*

A **probability distribution** describes the probabilities of all the possible outcomes for a random variable. The probabilities of all possible outcomes must sum to 1. A simple probability distribution is that for the roll of one fair die; there are six possible outcomes and each one has a probability of 1/6, so they sum to 1. The probability distribution of all the possible returns on the S&P 500 Index for the next year is a more complex version of the same idea.

A **discrete random variable** is one for which the number of possible outcomes can be counted, and for each possible outcome, there is a measurable and positive probability. An example of a discrete random variable is the number of days it will rain in a given month, because there is a countable number of possible outcomes, ranging from zero to the number of days in the month.

A **probability function**, denoted p(x), specifies the probability that a random variable is equal to a specific value. More formally, p(x) is the probability that random variable *X* takes

on the value *x*, or p(x) = P(X = x).

The two key properties of a probability function are:

- $0 \le p(x) \le 1$.
- $\sum p(x) = 1$, the sum of the probabilities for *all* possible outcomes, *x*, for a random variable, *X*, equals 1.

**EXAMPLE: Evaluating a probability function**

Consider the following function: $X = \{1, 2, 3, 4\}$, $p(x) = \frac{x}{10}$, else $p(x) = 0$

Determine whether this function satisfies the conditions for a probability function.

**Answer:**

Note that all of the probabilities are between 0 and 1, and the sum of all probabilities equals 1:

$$\sum p\left(x\right) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = 0.1 + 0.2 + 0.3 + 0.4 = 1$$

Both conditions for a probability function are satisfied.

A **continuous random variable** is one for which the number of possible outcomes is infinite, even if lower and upper bounds exist. The actual amount of daily rainfall between zero and 100 inches is an example of a continuous random variable because the actual amount of rainfall can take on an infinite number of values. Daily rainfall can be measured in inches, half inches, quarter inches, thousandths of inches, or even smaller increments. Thus, the number of possible daily rainfall amounts between zero and 100 inches is essentially infinite.

The assignment of probabilities to the possible outcomes for discrete and continuous random variables provides us with discrete probability distributions and continuous probability distributions. The difference between these types of distributions is most apparent for the following properties:

- For a *discrete distribution*, p(x) = 0 when x cannot occur, or p(x) > 0 if it can. Recall that p(x) is read: "the probability that random variable X = x." For example, the probability of it raining on 33 days in June is zero because this cannot occur, but the probability of it raining 25 days in June has some positive value.
- For a *continuous distribution*, p(x) = 0 even though *x* can occur. We can only consider $P(x_1 \le X \le x_2)$ where $x_1$ and $x_2$ are actual numbers. For example, the probability of receiving two inches of rain in June is zero because two inches is a single point in an infinite range of possible values. On the other hand, the probability of the amount of rain being between 1.99999999 and 2.00000001 inches has some positive value. In the case of continuous distributions, $P(x_1 \le X \le x_2) = P(x_1 < X < x_2)$ because $p(x_1) = p(x_2) = 0$.

In finance, some discrete distributions are treated as though they are continuous because the number of possible outcomes is very large. For example, the increase or decrease in the price of a stock traded on an American exchange is recorded in dollars and cents. Yet, the probability of a change of exactly \$1.33 or \$1.34 or any other specific change is almost zero. It is customary, therefore, to speak in terms of the probability of a range of possible price change, say between \$1.00 and \$2.00. In other words p(price change = 1.33) is essentially zero, but p(\$1 < price change < \$2) is greater than zero.

## LOS 9.c: Interpret a cumulative distribution function.

## LOS 9.d: Calculate and interpret probabilities for a random variable, given its cumulative distribution function.

*CFA® Program Curriculum, Volume 1, page 516*

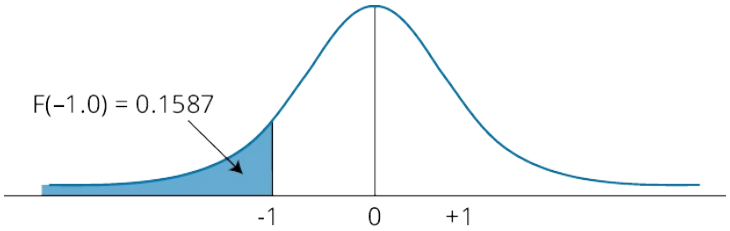
A **cumulative distribution function** (cdf), or simply *distribution function*, defines the probability that a random variable, *X*, takes on a value equal to or less than a specific value, *x*. It represents the sum, or *cumulative value*, of the probabilities for the outcomes up to and including a specified outcome. The cumulative distribution function for a random variable, *X*, may be expressed as $F(x) = P(X \leq x)$.

Consider the probability function defined earlier for X = {1, 2, 3, 4}, p(x) = x / 10. For this distribution, F(3) = 0.6 = 0.1 + 0.2 + 0.3, and F(4) = 1 = 0.1 + 0.2 + 0.3 + 0.4. This means that F(3) is the cumulative probability that outcomes 1, 2, or 3 occur, and F(4) is the cumulative probability that one of the possible outcomes occurs.

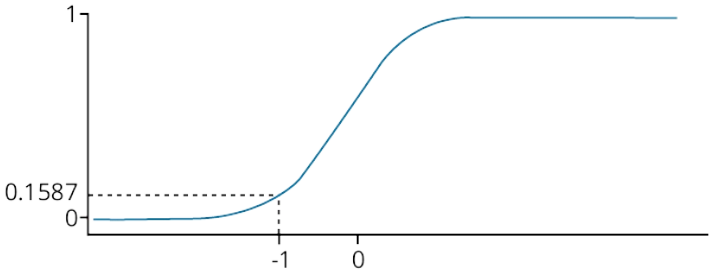
Figure 9.1 shows an example of a cumulative distribution function (for a standard normal distribution, described later in this topic). There is a 15.87% probability of a value less than –1. This is the total area to the left of –1 in the pdf in Panel (a), and the y-axis value of the cdf for a value of –1 in Panel (b).

**Figure 9.1: Standard Normal Probability Density and Cumulative Distribution Functions**

**(a) Probability density function**




**(b) Cumulative distribution function**




**EXAMPLE: Cumulative distribution function**

Return on equity for a firm is defined as a continuous distribution over the range from –20% to +30% and has a cumulative distribution function of F(x) = (x + 20) / 50. Calculate the probability that ROE will be between 0% and 15%.

**Answer:**

To determine the probability that ROE will be between 0% and 15%, we can first calculate the probability that ROE will be less than or equal to 15%, or F(15), and then subtract the probability that ROE will be less than zero, or F(0).

$P(0 \le x \le 15) = F(15) - F(0)$

$F(15) = (15 + 20) / 50 = 0.70$

$F(0) = (0 + 20) / 50 = 0.40$

$F(15) - F(0) = 0.70 - 0.40 = 0.30 = 30\%$

## LOS 9.e: Define a discrete uniform random variable, a Bernoulli random variable, and a binomial random variable.

## LOS 9.f: Calculate and interpret probabilities given the discrete uniform and the binomial distribution functions.

*CFA® Program Curriculum, Volume 1, page 516*

A **discrete uniform random variable** is one for which the probabilities for all possible outcomes for a discrete random variable are equal. For example, consider the *discrete uniform probability distribution* defined as X = {1, 2, 3, 4, 5}, p(x) = 0.2. Here, the probability for each outcome is equal to 0.2 [i.e., p(1) = p(2) = p(3) = p(4) = p(5) = 0.2]. Also, the cumulative distribution function for the *n*th outcome, $F(x_n) = np(x)$, and the probability for a range of outcomes is p(x)k, where *k* is the number of possible outcomes in the range.

**EXAMPLE: Discrete uniform distribution**

Determine p(6), F(6), and $P(2 \le X \le 8)$ for the discrete uniform distribution function defined as:

X = {2, 4, 6, 8, 10}, p(x) = 0.2

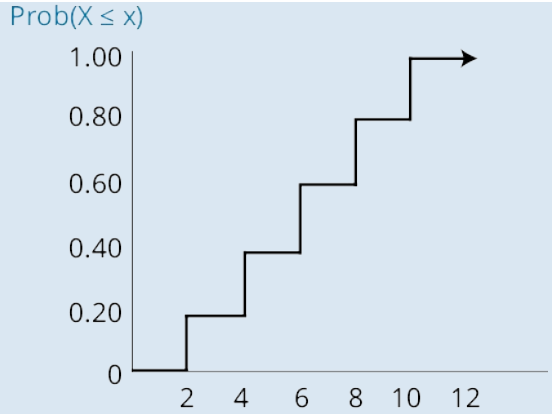
**Answer:**

p(6) = 0.2, since p(x) = 0.2 for all *x*. $F(6) = P(X \le 6) = np(x) = 3(0.2) = 0.6$. Note that n = 3 since 6 is the third outcome in the range of possible outcomes. $P(2 \le X \le 8) = 4(0.2) = 0.8$. Note that k = 4, since there are four outcomes in the range $2 \le X \le 8$. The following figures illustrate the concepts of a probability function and cumulative distribution function for this distribution.

**Probability and Cumulative Distribution Functions**

| X = x | Probability of x Prob (X = x) | Cumulative Distribution Function Prob (X < x) |
|---|---|---|
| 2 | 0.20 | 0.20 |
| 4 | 0.20 | 0.40 |
| 6 | 0.20 | 0.60 |
| 8 | 0.20 | 0.80 |

**Cumulative Distribution Function for X ~ Uniform {2, 4, 6, 8, 10}**




## The Binomial Distribution

A **binomial random variable** may be defined as the number of "successes" in a given number of trials, whereby the outcome can be either "success" or "failure." The probability of success, *p*, is constant for each trial, and the trials are independent. A binomial random variable for which the number of trials is 1 is called a **Bernoulli random variable**. Think of a trial as a mini-experiment (or "Bernoulli trial"). The final outcome is the number of successes in a series of *n* trials. Under these conditions, the binomial probability function defines the probability of *x* successes in *n* trials. It can be expressed using the following formula:

$p(x) = P(X = x) = (\text{number of ways to choose } x \text{ from } n)p^x(1 - p)^{n-x}$

where:

(number of ways to choose *x* from *n*) = $\frac{n!}{(n-x)!x!}$ which may also be denoted as $\binom{n}{x}$ or stated as "*n* choose *x*"

p = the probability of "success" on each trial [don't confuse it with p(x)]

So the probability of exactly *x* successes in *n* trials is:

$$p(x) = \frac{n!}{(n-x)!x!}p^x(1-p)^{n-x}$$

**EXAMPLE: Binomial probability**

Assuming a binomial distribution, compute the probability of drawing three black beans from a bowl of black and white beans if the probability of selecting a black bean in any given attempt is 0.6. You will draw five beans from the bowl.

**Answer:**

$P(X = 3) = p(3) = \frac{5!}{2!3!}(0.6)^3(0.4)^2 = (120 / 12)(0.216)(0.160) = 0.3456$

Some intuition about these results may help you remember the calculations. Consider that a (very large) bowl of black and white beans has 60% black beans and that each time you select a bean, you replace it in the bowl before drawing again. We want to know the probability of selecting exactly three black beans in five draws, as in the previous example.

One way this might happen is BBBWW. Since the draws are independent, the probability of

this is easy to calculate. The probability of drawing a black bean is 60%, and the probability of drawing a white bean is 1 − 60% = 40%. Therefore, the probability of selecting BBBWW, in order, is 0.6 × 0.6 × 0.6 × 0.4 × 0.4 = 3.456%. This is the $p^3(1 - p)^2$ from the formula and *p* is 60%, the probability of selecting a black bean on any single draw from the bowl. BBBWW is not, however, the only way to choose exactly three black beans in five trials. Another possibility is BBWWB, and a third is BWWBB. Each of these will have exactly the same probability of occurring as our initial outcome, BBBWW. That's why we need to answer the question of how many ways (different orders) there are for us to choose three black beans in five draws. Using the formula, there are $\frac{5!}{3!(5-3)!} = 10$ ways; 10 × 3.456% = 34.56%, the answer we computed above.

## Expected Value and Variance of a Binomial Random Variable

For a given series of *n* trials, the expected number of successes, or E(X), is given by the following formula:

expected value of X = E(X) = np

The intuition is straightforward; if we perform *n* trials and the probability of success on each trial is *p*, we expect *np* successes.

The variance of a binomial random variable is given by:

variance of X = np(1 − p)

**EXAMPLE: Expected value of a binomial random variable**

Based on empirical data, the probability that the Dow Jones Industrial Average (DJIA) will increase on any given day has been determined to equal 0.67. Assuming that the only other outcome is that it decreases, we can state p(UP) = 0.67 and p(DOWN) = 0.33. Further, assume that movements in the DJIA are independent (i.e., an increase in one day is independent of what happened on another day).

Using the information provided, compute the expected value of the number of up days in a 5-day period.

**Answer:**

Using binomial terminology, we define success as UP, so p = 0.67. Note that the definition of success is critical to any binomial problem.

E(X | n = 5, p = 0.67) = (5)(0.67) = 3.35

Recall that the "|" symbol means *given*. Hence, the preceding statement is read as: the expected value of X given that n = 5, and the probability of success = 67% is 3.35.

We should note that since the binomial distribution is a discrete distribution, the result X = 3.35 is not possible. However, if we were to record the results of many 5-day periods, the average number of up days (successes) would converge to 3.35.

### LOS 9.g: Construct a binomial tree to describe stock price movement.

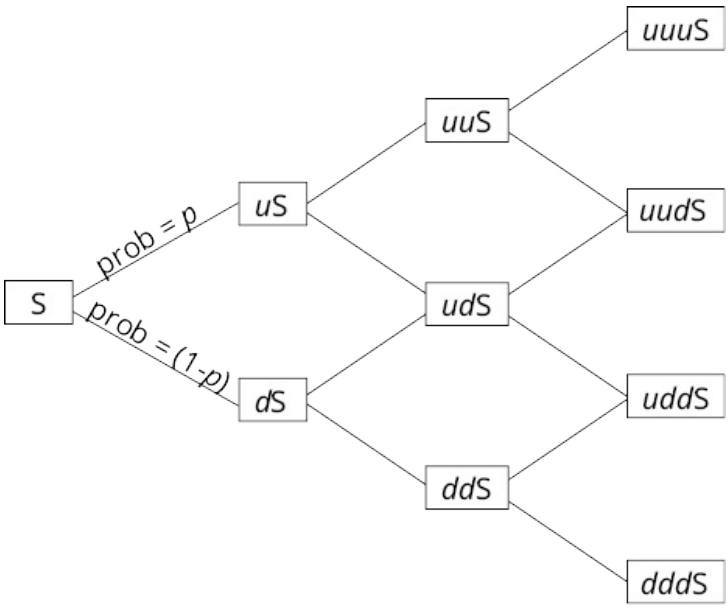
*CFA® Program Curriculum, Volume 1, page 526*

A binomial model can be applied to stock price movements. We just need to define the two possible outcomes and the probability that each outcome will occur. Consider a stock with current price S that will, over the next period, either increase in value or decrease in value (the only two possible outcomes). The probability of an up-move (the **up transition**

**probability**, *u*) is *p* and the probability of a down-move (the **down transition probability**, *d*) is (1 − *p*).

A **binomial tree** is constructed by showing all the possible combinations of up-moves and down-moves over a number of successive periods. For two periods, these combinations are *uu*, *ud*, *du*, and *dd*. Each of the possible values along a binomial tree is a **node**. Figure 9.2 illustrates a binomial tree for three periods.

**Figure 9.2: A Binomial Tree**



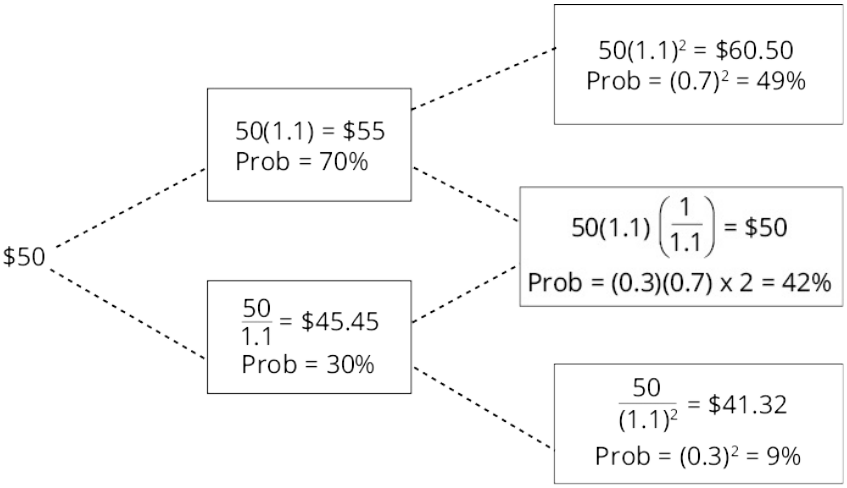

With an initial stock price S = 50, *u* = 1.01, *d* = $1/_{1.01}$, and Prob(*u*) = 0.6, we can calculate the possible stock prices after two periods as:

$$
\begin{aligned}
uuS &= 1.01^2 \times 50 = 51.01 \text{ with probability } (0.6)^2 = 0.36 \\
udS &= 1.01\,(1/_{1.01}) \times 50 = 50 \text{ with probability } (0.6)(0.4) = 0.24 \\
duS &= (1/_{1.01})\,(1.01) \times 50 = 50 \text{ with probability } (0.4)(0.6) = 0.24 \\
ddS &= (1/_{1.01})^2 \times 50 = 49.01 \text{ with probability } (0.4)^2 = 0.16
\end{aligned}
$$

Since a stock price of 50 can result from either *ud* or *du* moves, the probability of a stock price of 50 after two periods (the middle value) is 2 × (0.6)(0.4) = 48%.

A binomial tree with S = 50, *u* = 1.1, and Prob(*u*) = 0.7 is illustrated in Figure 9.3. Note that the middle value after two periods (50) is equal to the beginning value because *d* = 1 / *u*, a special case of the binomial model. The probability that the stock price is down (<50) after two periods is simply the probability of two down movements, $(1 - 0.7)^2$ = 9%.

**Figure 9.3: A Two-Period Binomial Tree S = $50, U = 1.10, Prob(U) = 0.7**




One of the important applications of a binomial stock price model is in pricing options. We can make a binomial tree for asset prices more realistic by shortening the length of the periods and increasing the number of periods and possible outcomes.

### LOS 9.h: Define the continuous uniform distribution and calculate and interpret probabilities, given a continuous uniform distribution.

*CFA® Program Curriculum, Volume 1, page 528*

The **continuous uniform distribution** is defined over a range that spans between some lower limit, *a*, and some upper limit, *b*, which serve as the parameters of the distribution. Outcomes can only occur between *a* and *b*, and since we are dealing with a continuous distribution, even if $a < x < b$, $P(X = x) = 0$. Formally, the properties of a continuous uniform distribution may be described as follows:

- For all $a \le x_1 < x_2 \le b$ (i.e., for all $x_1$ and $x_2$ between the boundaries *a* and *b*).
- $P(X < a \text{ or } X > b) = 0$ (i.e., the probability of *X* outside the boundaries is zero).
- $P(x_1 \le X \le x_2) = (x_2 - x_1)/(b - a)$. This defines the probability of outcomes between $x_1$ and $x_2$.

Don't miss how simple this is just because the notation is so mathematical. For a continuous uniform distribution, the probability of outcomes in a range that is one-half the whole range is 50%. The probability of outcomes in a range that is one-quarter as large as the whole possible range is 25%.

**EXAMPLE: Continuous uniform distribution**

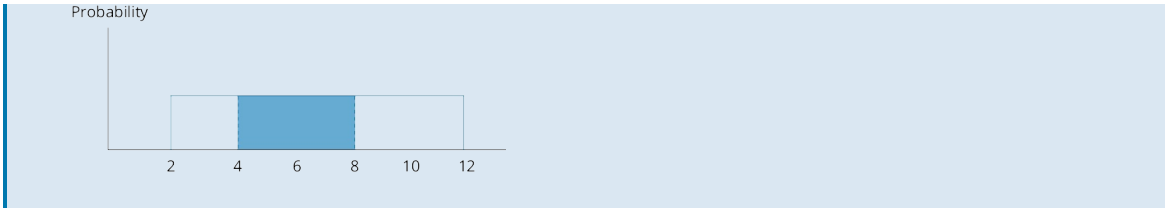
*X* is uniformly distributed between 2 and 12. Calculate the probability that *X* will be between 4 and 8.

**Answer:**

$$\frac{8-4}{12-2} = \frac{4}{10} = 40\%$$

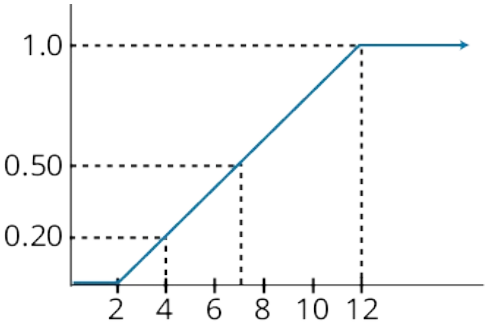
The figure below illustrates this continuous uniform distribution. Note that the area bounded by 4 and 8 is 40% of the total probability between 2 and 12 (which is 100%).

**Continuous Uniform Distribution**




Since outcomes are equal over equal-size possible intervals, the cdf is linear over the variable's range. The cdf for the distribution in the example, Prob (X < x), is shown in Figure 9.4.

**Figure 9.4: CDF for a Continuous Uniform Variable**




## MODULE QUIZ 9.1

*To best evaluate your performance, enter your quiz answers online.*

1. Which of the following is *least likely* an example of a discrete random variable?
   A. The number of stocks a person owns.
   B. The time spent by a portfolio manager with a client.
   C. The number of days it rains in a month in Iowa City.
2. For a continuous random variable *X*, the probability of any single value of *X* is:
   A. one.
   B. zero.
   C. determined by the cdf.
3. Which of the following is *least likely* a probability distribution?
   A. X = [1,2,3,4]; Prob [$X_i$] = $\frac{X_i^2}{30}$.
   B. X = [5,10]; Prob [$X_i$] = $\frac{8-X_i}{5}$.
   C. X = [5,10]; Prob [$X_i$] = $\frac{X_i-3}{9}$.

**Use the following table to answer Questions 4 through 8.**

| Probability distribution of a discrete random variable X | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| X | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| P(X) | 0.04 | 0.11 | 0.18 | 0.24 | 0.14 | 0.17 | 0.09 | 0.03 |

4. The probability that X = 3 is:
   A. 0.18.
   B. 0.24.
   C. 0.43.
5. The cdf of 5, or F(5) is:
   A. 0.17.

B. 0.71.
C. 0.88.

6. The probability that *X* is *greater* than 3 is:
   A. 0.24.
   B. 0.43.
   C. 0.67.
7. What is $P(2 \le X \le 5)$?
   A. 0.17.
   B. 0.38.
   C. 0.73.
8. The expected value of the random variable *X* is:
   A. 3.35.
   B. 3.70.
   C. 5.47.
9. Which of the following is *least likely* a condition of a binomial experiment?
   A. There are only two trials.
   B. The trials are independent.
   C. If *p* is the probability of success, and *q* is the probability of failure, then $p + q = 1$.
10. Which of the following statements *least accurately* describes the binomial distribution?
    A. It is a discrete distribution.
    B. The probability of an outcome of zero is zero.
    C. The combination formula is used in computing probabilities.
11. A recent study indicated that 60% of all businesses have a fax machine. From the binomial probability distribution table, the probability that exactly four businesses will have a fax machine in a random selection of six businesses is:
    A. 0.138.
    B. 0.276.
    C. 0.311.
12. Ten percent of all college graduates hired stay with the same company for more than five years. In a random sample of six recently hired college graduates, the probability that exactly two will stay with the same company for more than five years is *closest* to:
    A. 0.098.
    B. 0.114.
    C. 0.185.
13. Assume that 40% of candidates who sit for the CFA® examination pass it the first time. Of a random sample of 15 candidates who are sitting for the exam for the first time, what is the expected number of candidates that will pass?
    A. 0.375.
    B. 4.000.
    C. 6.000.
14. A continuous uniform distribution has the parameters $a = 4$ and $b = 10$. The $F(20)$ is:
    A. 0.25.
    B. 0.50.
    C. 1.00.

## MODULE 9.2: NORMAL DISTRIBUTIONS

Video covering this content is available online.

### LOS 9.i: Explain the key properties of the normal distribution.

*CFA® Program Curriculum, Volume 1, page 531*

The normal distribution is important for many reasons. Many of the random variables that are relevant to finance and other professional disciplines follow a normal distribution. In the area

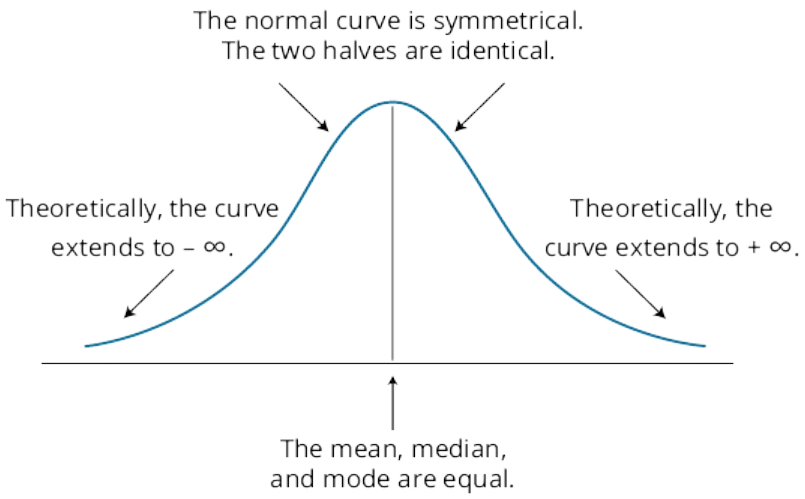
of investment and portfolio management, the normal distribution plays a central role in portfolio theory.

The **normal distribution** has the following key properties:

- It is completely described by its mean, $\mu$, and variance, $\sigma^2$, stated as $X \sim N(\mu, \sigma^2)$. In words, this says that "$X$ is normally distributed with mean $\mu$ and variance $\sigma^2$."
- Skewness = 0, meaning that the normal distribution is symmetric about its mean, so that $P(X \leq \mu) = P(\mu \leq X) = 0.5$, and mean = median = mode.
- Kurtosis = 3; this is a measure of how flat the distribution is. Recall that excess kurtosis is measured relative to 3, the kurtosis of the normal distribution.
- A linear combination of normally distributed random variables is also normally distributed.
- The probabilities of outcomes further above and below the mean get smaller and smaller but do not go to zero (the tails get very thin but extend infinitely).

Many of these properties are evident from examining the graph of a normal distribution's probability density function as illustrated in Figure 9.5.

**Figure 9.5: Normal Distribution Probability Density Function**




## LOS 9.j: Distinguish between a univariate and a multivariate distribution and explain the role of correlation in the multivariate normal distribution.

*CFA® Program Curriculum, Volume 1, page 532*

Up to this point, our discussion has been strictly focused on **univariate distributions** (i.e., the distribution of a single random variable). In practice, however, the relationships between two or more random variables are often relevant. For instance, investors and investment managers are frequently interested in the interrelationship among the returns of one or more assets. In fact, as you will see in your study of asset pricing models and modern portfolio theory, the return on a given stock and the return on the S&P 500 or some other market index will have special significance. Regardless of the specific variables, the simultaneous analysis of two or more random variables requires an understanding of multivariate distributions.

A **multivariate distribution** specifies the probabilities associated with a group of random variables and is meaningful only when the behavior of each random variable in the group is in some way dependent upon the behavior of the others. Both discrete and continuous random variables can have multivariate distributions. Multivariate distributions between two discrete random variables are described using joint probability tables. For continuous random variables, a multivariate *normal* distribution may be used to describe them if all of the individual variables follow a normal distribution. As previously mentioned, one of the characteristics of a normal distribution is that a linear combination of normally distributed random variables is normally distributed as well. For example, if the return of each stock in a portfolio is normally distributed, the return on the portfolio will also be normally distributed.

## The Role of Correlation in the Multivariate Normal Distribution

Similar to a univariate normal distribution, a multivariate normal distribution can be described by the mean and variance of the individual random variables. Additionally, it is necessary to specify the correlation between the individual pairs of variables when describing a multivariate distribution. Correlation is the feature that distinguishes a multivariate distribution from a univariate normal distribution. Correlation indicates the strength of the linear relationship between a pair of random variables.

Using asset returns as our random variables, the multivariate normal distribution for the returns on *n* assets can be completely defined by the following three sets of parameters:

- *n* means of the *n* series of returns ($\mu_1, \mu_2, \ldots, \mu_n$).
- *n* variances of the *n* series of returns ($\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$).
- 0.5n(n − 1) pair-wise correlations.

For example, if there are two assets, n = 2, then the multivariate returns distribution can be described with two means, two variances, and one correlation [0.5(2)(2 − 1) = 1]. If there are four assets, n = 4, the multivariate distribution can be described with four means, four variances, and six correlations [0.5(4)(4 − 1) = 6]. When building a portfolio of assets, all other things being equal, it is desirable to combine assets having low returns correlation because this will result in a portfolio with a lower variance than one composed of assets with higher correlations.

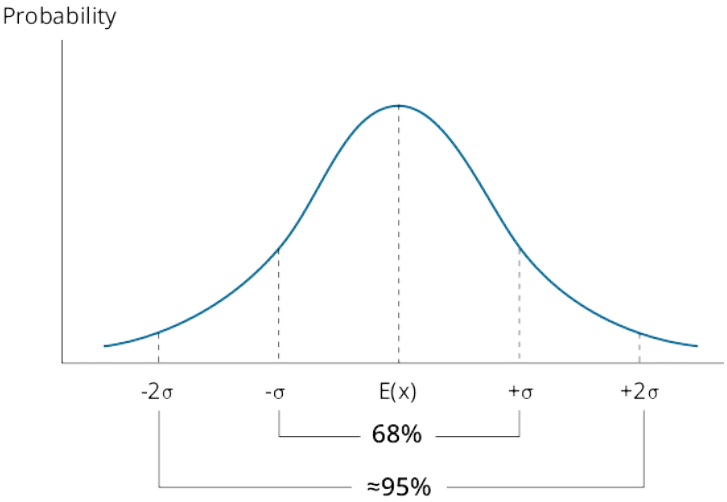
**LOS 9.k: Determine the probability that a normally distributed random variable lies inside a given interval.**

*CFA® Program Curriculum, Volume 1, page 533*

A **confidence interval** is a range of values around the expected outcome within which we expect the actual outcome to be some specified percentage of the time. A 95% confidence interval is a range that we expect the random variable to be in 95% of the time. For a normal distribution, this interval is based on the expected value (sometimes called a point estimate) of the random variable and on its variability, which we measure with standard deviation.

Confidence intervals for a normal distribution are illustrated in Figure 9.6. For any normally distributed random variable, 68% of the outcomes are within one standard deviation of the expected value (mean), and approximately 95% of the outcomes are within two standard deviations of the expected value.

**Figure 9.6: Confidence Intervals for a Normal Distribution**




In practice, we will not know the actual values for the mean and standard deviation of the distribution, but will have estimated them as $\overline{X}$ and *s*. The three confidence intervals of most interest are given by:

- The 90% confidence interval for X is $\overline{X} - 1.65s$ to $\overline{X} + 1.65s$.
- The 95% confidence interval for X is $\overline{X} - 1.96s$ to $\overline{X} + 1.96s$.
- The 99% confidence interval for X is $\overline{X} - 2.58s$ to $\overline{X} + 2.58s$.

**EXAMPLE: Confidence intervals**

The average return of a mutual fund is 10.5% per year and the standard deviation of annual returns is 18%. If returns are approximately normal, what is the 95% confidence interval for the mutual fund return next year?

**Answer:**

Here $\mu$ and $\sigma$ are 10.5% and 18%, respectively. Thus, the 95% confidence interval for the return, *R*, is:

$$10.5 \pm 1.96(18) = -24.78\% \text{ to } 45.78\%$$

Symbolically, this result can be expressed as:

$$P(-24.78 < R < 45.78) = 0.95 \text{ or } 95\%$$

The interpretation is that the annual return is expected to be within this interval 95% of the time, or 95 out of 100 years.

## LOS 9.l: Define the standard normal distribution, explain how to standardize a random variable, and calculate and interpret probabilities using the standard normal distribution.

*CFA® Program Curriculum, Volume 1, page 534*

The **standard normal distribution** is a normal distribution that has been standardized so that it has a mean of zero and a standard deviation of 1 [i.e., N~(0,1)]. To standardize an observation from a given normal distribution, the *z-value* of the observation must be calculated. The *z*-value represents the number of standard deviations a given observation is from the population mean. *Standardization* is the process of converting an observed value for

a random variable to its *z*-value. The following formula is used to standardize a random variable:

$$z = \frac{\text{observation} - \text{population mean}}{\text{standard deviation}} = \frac{x - \mu}{\sigma}$$

**PROFESSOR'S NOTE**

The term *z-value* will be used for a standardized observation in this document. The terms *z-score* and *z-statistic* are also commonly used.

**EXAMPLE: Standardizing a random variable (calculating *z*-values)**

Assume that the annual earnings per share (EPS) for a population of firms are normally distributed with a mean of $6 and a standard deviation of $2.

What are the *z*-values for EPS of $2 and $8?

**Answer:**

If EPS = x = $8, then z = (x − μ) / σ = ($8 − $6) / $2 = +1

If EPS = x = $2, then z = (x − μ) / σ = ($2 − $6) / $2 = –2

Here, z = +1 indicates that an EPS of $8 is one standard deviation above the mean, and z = –2 means that an EPS of $2 is two standard deviations below the mean.

## Calculating Probabilities Using z-Values

Now we will show how to use standardized values (*z*-values) and a table of probabilities for Z to determine probabilities. A portion of a table of the cumulative distribution function for a standard normal distribution is shown in Figure 9.7. We will refer to this table as the *z*-table, as it contains values generated using the cumulative density function for a standard normal distribution, denoted by F(Z). Thus, the values in the *z*-table are the probabilities of observing a *z*-value that is less than a given value, *z* [i.e., P(Z < *z*)]. The numbers in the first column are *z*-values that have only one decimal place. The columns to the right supply probabilities for *z*-values with two decimal places.

Note that the *z*-table in Figure 9.7 only provides probabilities for positive *z*-values. This is not a problem because we know from the symmetry of the standard normal distribution that F(–Z) = 1 − F(Z). The tables in the back of many texts actually provide probabilities for negative *z*-values, but we will work with only the positive portion of the table because this may be all you get on the exam. In Figure 9.7, we can find the probability that a standard normal random variable will be less than 1.66, for example. The table value is 95.15%. The probability that the random variable will be less than –1.66 is simply 1 − 0.9515 = 0.0485 = 4.85%, which is also the probability that the variable will be greater than +1.66.

**Figure 9.7: Cumulative Probabilities for a Standard Normal Distribution**

| Cdf Values for the Standard Normal Distribution: The z-Table | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **z** | **.00** | **.01** | **.02** | **.03** | **.04** | **.05** | **.06** | **.07** | **.08** | **.09** |
| **0.0** | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| **0.1** | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| **0.2** | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| **0.5** | .6915 | Please note that several of the rows have been deleted to save space.* | | | | | | | | |
| **1.2** | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **1.6** | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| **1.8** | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| **1.9** | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| **2.0** | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| **2.5** | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| **3.0** | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |

*A complete cumulative standard normal table is included in Appendix A.

### Professor's Note

When you use the standard normal probabilities, you have formulated the problem in terms of standard deviations from the mean. Consider a security with returns that are approximately normal, an expected return of 10%, and standard deviation of returns of 12%. The probability of returns greater than 30% is calculated based on the number of standard deviations that 30% is above the expected return of 10%. 30% is 20% above the expected return of 10%, which is 20 / 12 = 1.67 standard deviations above the mean. We look up the probability of returns less than 1.67 standard deviations above the mean (0.9525 or 95.25% from Figure 9.7) and calculate the probability of returns more than 1.67 standard deviations above the mean as 1 – 0.9525 = 4.75%.

**EXAMPLE: Using the *z*-table (1)**

Considering again EPS distributed with μ = \$6 and σ = \$2, what is the probability that EPS will be \$9.70 or more?

**Answer:**

Here we want to know P(EPS > \$9.70), which is the area under the curve to the right of the *z*-value corresponding to EPS = \$9.70 (see the following figure).

The *z*-value for EPS = \$9.70 is:

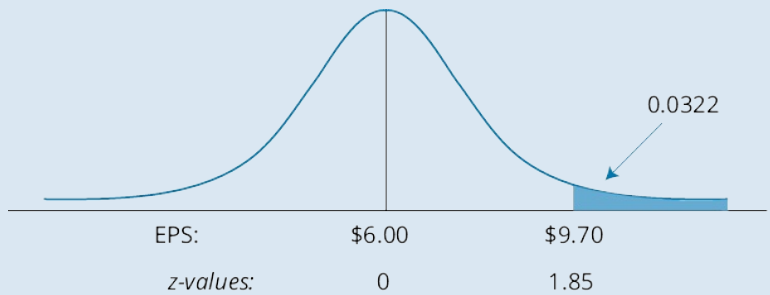
$$z = \frac{(x-\mu)}{\sigma} = \frac{(9.70-6)}{2} = 1.85$$

That is, \$9.70 is 1.85 standard deviations above the mean EPS value of \$6.

From the *z*-table we have F(1.85) = 0.9678, but this is P(EPS ≤ 9.70). We want P(EPS > 9.70), which is 1 – P(EPS ≤ 9.70).

P(EPS > 9.70) = 1 – 0.9678 = 0.0322, or 3.2%

**P(EPS > \$9.70)**




**EXAMPLE: Using the *z*-table (2)**

Using the distribution of EPS with μ = \$6 and σ = \$2 again, what percent of the observed EPS values are likely to be less than \$4.10?

**Answer:**

As shown graphically in the following figure, we want to know P(EPS < \$4.10). This requires a 2-step approach like the one taken in the preceding example.

First, the corresponding $z$-value must be determined as follows:

$$z = \frac{(\$4.10 - \$6)}{2} = -0.95,$$

so \$4.10 is 0.95 standard deviations below the mean of \$6.00.

Now, from the $z$-table for negative values in the back of this book, we find that F(–0.95) = 0.1711, or 17.11%.

**Finding a Left-Tail Probability**

The $z$-table gives us the probability that the outcome will be more than 0.95 standard deviations below the mean.

## MODULE QUIZ 9.2

*To best evaluate your performance, enter your quiz answers online.*

1. A key property of a normal distribution is that it:
   A. has zero skewness.
   B. is asymmetrical.
   C. has zero kurtosis.
2. Which of the following parameters is necessary to describe a multivariate normal distribution?
   A. Beta.
   B. Correlation.
   C. Degrees of freedom.

**Use the following table to answer Question 3.**

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 |
|---|---|---|---|---|---|
| **1.0** | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 |
| **1.1** | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 |
| **1.2** | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 |

3. A study of hedge fund investors found that their annual household incomes are normally distributed with a mean of \$175,000 and a standard deviation of \$25,000. The percentage of hedge fund investors that have incomes greater than \$150,000 is *closest* to:
   A. 34.13%.
   B. 68.26%.
   C. 84.13%.
4. For the standard normal distribution, the $z$-value gives the distance between the mean and a point in terms of:
   A. the variance.
   B. the standard deviation.
   C. the center of the curve.
5. For a standard normal distribution, F(0) is:
   A. 0.0.

B. 0.1.
C. 0.5.

## MODULE 9.3: LOGNORMAL DISTRIBUTION, SIMULATIONS

Video covering this content is available online.

**LOS 9.m: Define shortfall risk, calculate the safety-first ratio, and select an optimal portfolio using Roy's safety-first criterion.**

*CFA® Program Curriculum, Volume 1, page 537*

Shortfall risk is the probability that a portfolio value or return will fall below a particular (target) value or return over a given time period.

**Roy's safety-first criterion** states that the optimal portfolio minimizes the probability that the return of the portfolio falls below some minimum acceptable level. This minimum acceptable level is called the **threshold level**. Symbolically, Roy's safety-first criterion can be stated as:

minimize $P(R_p < R_L)$

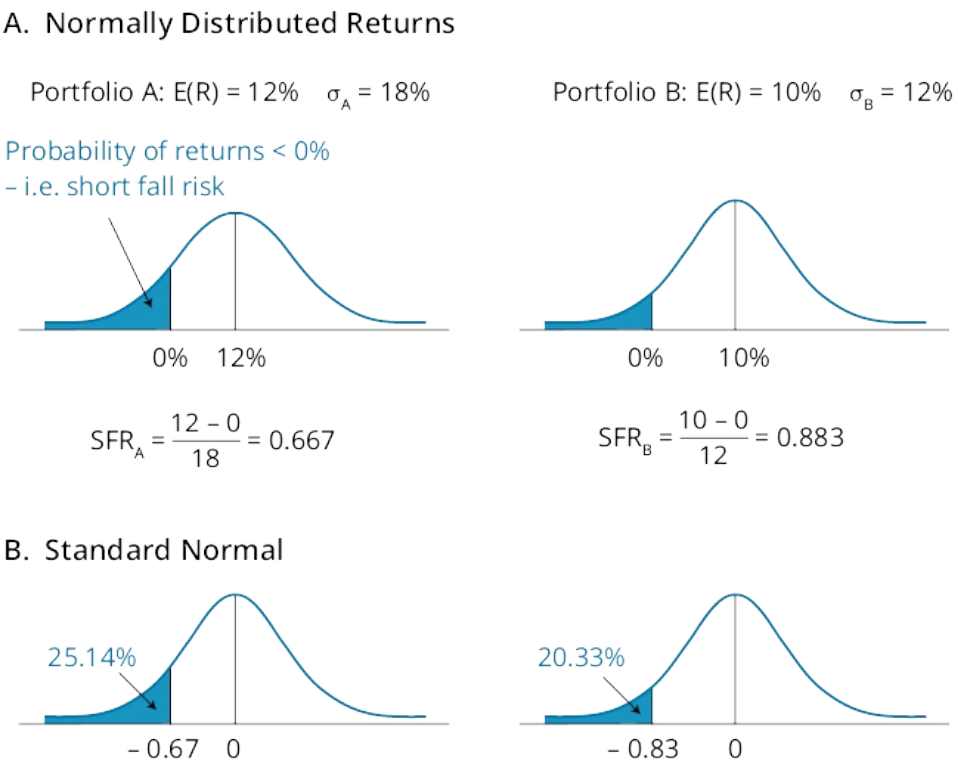
where:
$R_p$ = portfolio return
$R_L$ = threshold level return

If portfolio returns are normally distributed, then Roy's safety-first criterion can be stated as:

maximize the SFRatio, where SFRatio = $\frac{[E(R_p) - R_L]}{\sigma_p}$

The reasoning behind the safety-first criterion is illustrated in Figure 9.8. Assume an investor is choosing between two portfolios: Portfolio A with expected return of 12% and standard deviation of returns of 18%, and Portfolio B with expected return of 10% and standard deviation of returns of 12%. The investor has stated that he wants to minimize the probability of losing money (negative returns). Assuming that returns are normally distributed, the portfolio with the larger SFR using 0% as the threshold return ($R_L$) will be the one with the lower probability of negative returns.

**Figure 9.8: The Safety-First Criterion and Shortfall Risk**




Panel B of Figure 9.8 relates the SFRatio to the standard normal distribution. Note that the SFR is the number of standard deviations *below* the mean. Thus, the portfolio with the larger SFR has the lower probability of returns below the threshold return, which is a return of 0% in our example. Using a *z*-table for negative values, we can find the probabilities in the left-hand tails as indicated. These probabilities (25% for Portfolio A and 20% for Portfolio B) are also the shortfall risk for a target return of 0%, that is, the probability of negative returns. Portfolio B has the higher SFR which means it has the lower probability of negative returns.

In summary, when choosing among portfolios with normally distributed returns using Roy's safety-first criterion, there are two steps:

*Step 1:* Calculate the SFRatio = $\frac{[E(R_p) - R_L]}{\sigma_p}$.

*Step 2:* Choose the portfolio that has the *largest* SFRatio.

**EXAMPLE: Roy's safety-first criterion**

For the next year, the managers of a $120 million college endowment plan have set a minimum acceptable end-of-year portfolio value of $123.6 million. Three portfolios are being considered which have the expected returns and standard deviation shown in the first two rows of the following table. Determine which of these portfolios is the most desirable using Roy's safety-first criterion and the probability that the portfolio value will fall short of the target amount.

**Answer:**

The threshold return is $R_L$ = (123.6 − 120) / 120 = 0.030 = 3%. The SFRs are shown in the table below. As indicated, the best choice is Portfolio A because it has the largest SFR.

**Roy's Safety-First Ratios**

| Portfolio | Portfolio A | Portfolio B | Portfolio C |
|---|---|---|---|
| $E(R_p)$ | 9% | 11% | 6.6% |

| | | | |
|---|---|---|---|
| $\sigma_p$ | 12% | 20% | 8.2% |
| SFRatio | $0.5 = \frac{(9-3)}{12}$ | $0.4 = \frac{(11-3)}{20}$ | $0.44 = \frac{(6.6-3)}{8.2}$ |

The probability of an ending value for Portfolio A less than $123.6 million (a return less than 3%) is simply F(–0.5) which we can find on the *z*-table for negative values. The probability is 0.3085 = 30.85%.

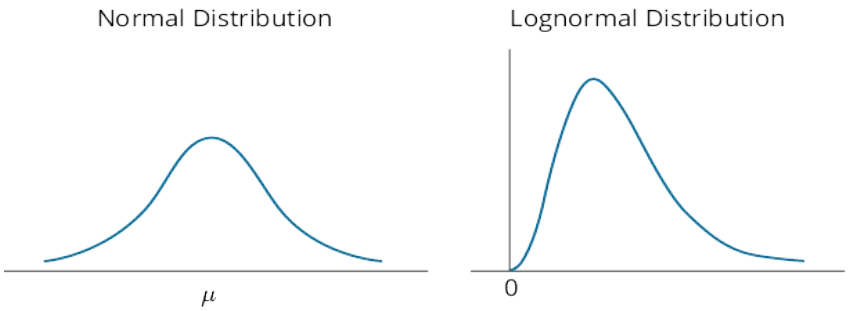
## LOS 9.n: Explain the relationship between normal and lognormal distributions and why the lognormal distribution is used to model asset prices.

*CFA® Program Curriculum, Volume 1, page 539*

The **lognormal distribution** is generated by the function $e^x$, where *x* is normally distributed. Since the natural logarithm, ln, of $e^x$ is *x*, the logarithms of lognormally distributed random variables are normally distributed, thus the name.

Figure 9.9 illustrates the differences between a normal distribution and a lognormal distribution.

**Figure 9.9: Normal vs. Lognormal Distributions**




In Figure 9.9, we can see that:

- The lognormal distribution is skewed to the right.
- The lognormal distribution is bounded from below by zero so that it is useful for modeling asset prices which never take negative values.

If we used a normal distribution of returns to model asset prices over time, we would admit the possibility of returns less than –100%, which would admit the possibility of asset prices less than zero. Using a lognormal distribution to model *price relatives* avoids this problem. A price relative is just the end-of-period price of the asset divided by the beginning price ($S_1/S_0$) and is equal to (1 + the holding period return). To get the end-of-period asset price, we can simply multiply the price relative times the beginning-of-period asset price. Since a lognormal distribution takes a minimum value of zero, end-of-period asset prices cannot be less than zero. A price relative of zero corresponds to a holding period return of –100% (i.e., the asset price has gone to zero). Recall that we used price relatives as the up-move and down-move (multiplier) terms in constructing a binomial tree for stock price changes over a number of periods.

## LOS 9.o: Distinguish between discretely and continuously compounded rates of return and calculate and interpret a continuously compounded rate of return, given a specific holding period return.

**Discretely compounded** returns are just the compound returns we are familiar with, given some discrete compounding period, such as semiannual or quarterly. Recall that the more frequent the compounding, the greater the effective annual return. For a stated rate of 10%, semiannual compounding results in an effective yield of $\left(1+\frac{0.10}{2}\right)^2-1=10.25\%$ and monthly compounding results in an effective yield of $\left(1+\frac{0.10}{12}\right)^{12}-1=10.47\%$. Daily or even hourly compounding will produce still larger effective yields. The limit of this exercise, as the compounding periods get shorter and shorter, is called **continuous compounding**. The effective annual rate, based on continuous compounding for a stated annual rate of $R_{cc}$, can be calculated from the formula:

$$\text{effective annual rate} = e^{R_{cc}} - 1$$

Based on a stated rate of 10%, the effective rate with continuous compounding is $e^{0.10} - 1 =$ 10.5171%. Please verify this by entering 0.1 in your calculator and finding the ex function.

Since the natural log, ln, of $e^x$ is *x*, we can get the continuously compounded rate from an effective annual rate by using the ln calculator function. Using our previous example, ln (1 + 10.517%) = ln 1.105171 = 10%. Verify this by entering 1.105171 in your calculator and then entering the ln key. (Using the HP calculator, the keystrokes are 1.105171 [g] [ln].)

We can use this method to find the continuously compounded rate that will generate a particular holding period return. If we are given a holding period return of 12.5% for the year, the equivalent continuously compounded rate is ln 1.125 = 11.778%. Since the calculation is based on 1 plus the holding period return, we can also do the calculation directly from the *price relative*. The price relative is just the end-of-period value divided by the beginning-of-period value. The continuously compounded rate of return is:

$$\ln\left(\frac{S_1}{S_0}\right) = \ln(1+\text{HPR}) = R_{cc}$$

**EXAMPLE: Calculating continuously compounded returns**

A stock was purchased for $100 and sold one year later for $120. Calculate the investor's annual rate of return on a continuously compounded basis.

**Answer:**

$$\ln\left(\frac{120}{100}\right) = 18.232\%$$

If we had been given the return (20%) instead, the calculation is:

ln(1 + 0.20) = 18.232%

One property of continuously compounded rates of return is that they are additive for multiple periods. Note that the (effective) holding period return over two years is calculated by doubling the continuously compounded annual rate. If $R_{cc}$ = 10%, the (effective) holding period return over two years is $e^{(0.10)2} - 1$ = 22.14%. In general, the holding period return after *T* years, when the annual continuously compounded rate is $R_{cc}$, is given by:

$$HPR_T = e^{R_{CC} \times T} - 1$$

Given investment results over a 2-year period, we can calculate the 2-year continuously compounded return and divide by two to get the annual rate. Consider an investment that appreciated from $1,000 to $1,221.40 over a 2-year period. The 2-year continuously compounded rate is ln(1,221.40 / 1,000) = 20%, and the annual continuously compounded rate ($R_{CC}$) is 20% / 2 = 10%.

### LOS 9.p: Explain Monte Carlo simulation and describe its applications and limitations.

*CFA® Program Curriculum, Volume 1, page 545*

**Monte Carlo simulation** is a technique based on the repeated generation of one or more risk factors that affect security values, in order to generate a distribution of security values. For each of the risk factors, the analyst must specify the parameters of the probability distribution that the risk factor is assumed to follow. A computer is then used to generate random values for each risk factor based on its assumed probability distributions. Each set of randomly generated risk factors is used with a pricing model to value the security. This procedure is repeated many times (100s, 1,000s, or 10,000s), and the distribution of simulated asset values is used to draw inferences about the expected (mean) value of the security and possibly the variance of security values about the mean as well.

As an example, consider the valuation of stock options that can only be exercised on a particular date. The main risk factor is the value of the stock itself, but interest rates could affect the valuation as well. The simulation procedure would be to:

1. Specify the probability distributions of stock prices and of the relevant interest rate, as well as the parameters (mean, variance, possibly skewness) of the distributions.
2. Randomly generate values for both stock prices and interest rates.
3. Value the options for each pair of risk factor values.
4. After many iterations, calculate the mean option value and use that as your estimate of the option's value.

Monte Carlo simulation is used to:

- Value complex securities.
- Simulate the profits/losses from a trading strategy.
- Calculate estimates of value at risk (VaR) to determine the riskiness of a portfolio of assets and liabilities.
- Simulate pension fund assets and liabilities over time to examine the variability of the difference between the two.
- Value portfolios of assets that have nonnormal returns distributions.

The **limitations of Monte Carlo simulation** are that it is fairly complex and will provide answers that are no better than the assumptions about the distributions of the risk factors and the pricing/valuation model that is used. Also, simulation is not an analytic method, but a statistical one, and cannot provide the insights that analytic methods can.

### LOS 9.q: Compare Monte Carlo simulation and historical simulation.

*CFA® Program Curriculum, Volume 1, page 550*

**Historical simulation** is based on actual changes in value or actual changes in risk factors over some prior period. Rather than model the distribution of risk factors, as in Monte Carlo simulation, the set of all changes in the relevant risk factors over some prior period is used. Each iteration of the simulation involves randomly selecting one of these past changes for each risk factor and calculating the value of the asset or portfolio in question, based on those changes in risk factors.

Historical simulation has the advantage of using the actual distribution of risk factors so that the distribution of changes in the risk factors does not have to be estimated. It suffers from the fact that past changes in risk factors may not be a good indication of future changes. Events that occur infrequently may not be reflected in historical simulation results unless the events occurred during the period from which the values for risk factors are drawn. An additional limitation of historical simulation is that it cannot address the sort of "what if" questions that Monte Carlo simulation can. With Monte Carlo simulation, we can investigate the effect on the distribution of security/portfolio values if we increase the variance of one of the risk factors by 20%; with historical simulation we cannot do this.

## MODULE QUIZ 9.3

*To best evaluate your performance, enter your quiz answers online.*

**Use the following table to answer Questions 1 and 2.**

| Portfolio | Portfolio A | Portfolio B | Portfolio C |
|---|---|---|---|
| $E(R_p)$ | 5% | 11% | 18% |
| $\sigma_p$ | 8% | 21% | 40% |

1. Given a threshold level of return of 4%, use Roy's safety-first criterion to choose the optimal portfolio. Portfolio:
   A. A.
   B. B.
   C. C.
2. Given a threshold level of return of 0%, use Roy's safety-first criterion to choose the optimal portfolio. Portfolio:
   A. A.
   B. B.
   C. C.
3. Portfolio A has a safety-first ratio of 1.3 with a threshold return of 2%. What is the shortfall risk for a threshold return of 2%?
   A. 9.68%.
   B. 40.30%.
   C. 90.30%.
4. For a lognormal distribution:
   A. the mean equals the median.
   B. the probability of a negative outcome is zero.
   C. the probability of a positive outcome is 50%.
5. If a stock's initial price is $20 and its year-end price is $23, then its continuously compounded annual rate of return is:
   A. 13.64%.
   B. 13.98%.
   C. 15.00%.
6. A stock doubled in value last year. Its continuously compounded return over the period was *closest* to:
   A. 18.2%.

B. 69.3%.
C. 100.0%.

7. Using hypothesized parameter values and a random number generator to study the behavior of certain asset returns is part of:
   A. historical analysis.
   B. Monte Carlo simulation.
   C. standardizing a random variable.

## KEY CONCEPTS

### LOS 9.a

A probability distribution lists all the possible outcomes of an experiment, along with their associated probabilities.

A discrete random variable has positive probabilities associated with a finite number of outcomes.

A continuous random variable has positive probabilities associated with a range of outcome values—the probability of any single value is zero.

### LOS 9.b

The set of possible outcomes of a specific discrete random variable is a finite set of values. An example is the number of days last week on which the value of a particular portfolio increased. For a discrete distribution, $p(x) = 0$ when $x$ cannot occur, or $p(x) > 0$ if it can.

### LOS 9.c

A cumulative distribution function (cdf) gives the probability that a random variable will be less than or equal to specific values. The cumulative distribution function for a random variable $X$ may be expressed as $F(x) = P(X \leq x)$.

### LOS 9.d

Given the cumulative distribution function for a random variable, the probability that an outcome will be less than or equal to a specific value is represented by the area under the probability distribution to the left of that value.

### LOS 9.e

A discrete uniform distribution is one where there are $n$ discrete, equally likely outcomes.

The binomial distribution is a probability distribution for a binomial (discrete) random variable that has two possible outcomes.

### LOS 9.f

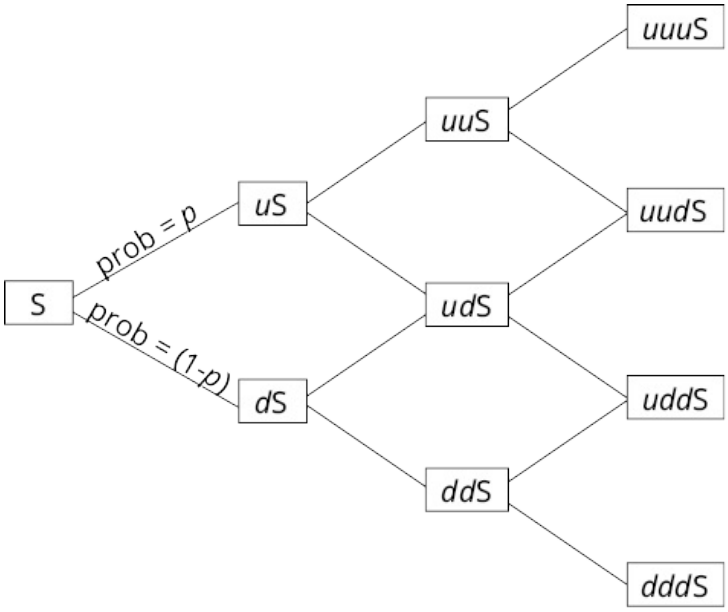
For a discrete uniform distribution with n possible outcomes, the probability for each outcome equals 1/n.

For a binomial distribution, if the probability of success is $p$, the probability of $x$ successes in $n$ trials is:

$$p(x) = P(X = x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} = {}_nC_x \times p^x (1-p)^{n-x}$$

### LOS 9.g

A binomial tree illustrates the probabilities of all the possible values that a variable (such as a stock price) can take on, given the probability of an up-move and the magnitude of an up-move (the up-move factor).




With an initial stock price S = 50, $u$ = 1.01, $d$ = $^1/_{1.01}$, and prob($u$) = 0.6, we can calculate the possible stock prices after two periods as:

$$\begin{aligned}
uuS &= 1.01^2 \times 50 = 51.01 \text{ with probability } (0.6)^2 = 0.36 \\
udS &= 1.01\,(^1/_{1.01}) \times 50 = 50 \text{ with probability } (0.6)(0.4) = 0.24 \\
duS &= (^1/_{1.01})\,(1.01) \times 50 = 50 \text{ with probability } (0.4)(0.6) = 0.24 \\
ddS &= (^1/_{1.01})^2 \times 50 = 49.01 \text{ with probability } (0.4)^2 = 0.16
\end{aligned}$$

## LOS 9.h

A continuous uniform distribution is one where the probability of $X$ occurring in a possible range is the length of the range relative to the total of all possible values. Letting $a$ and $b$ be the lower and upper limit of the uniform distribution, respectively, then for:

$$a \le x_1 \le x_2 \le b,\ P\,(x_1 \le X \le x_2) = \frac{(x_2 - x_1)}{(b - a)}$$

## LOS 9.i

The normal probability distribution and normal curve have the following characteristics:

- The normal curve is symmetrical and bell-shaped with a single peak at the exact center of the distribution.
- Mean = median = mode, and all are in the exact center of the distribution.
- The normal distribution can be completely defined by its mean and standard deviation because the skew is always zero and kurtosis is always 3.

## LOS 9.j

Multivariate distributions describe the probabilities for more than one random variable, whereas a univariate distribution is for a single random variable.

The correlation(s) of a multivariate distribution describes the relation between the outcomes of its variables relative to their expected values.

## LOS 9.k

A confidence interval is a range within which we have a given level of confidence of finding a point estimate (e.g., the 90% confidence interval for X is $\overline{X} - 1.65s$ to $\overline{X} + 1.65s$).

Confidence intervals for any normally distributed random variable are:

- 90%: μ ± 1.65 standard deviations.
- 95%: μ ± 1.96 standard deviations.
- 99%: μ ± 2.58 standard deviations.

The probability that a normally distributed random variable *X* will be within *A* standard deviations of its mean, μ, [i.e., $P(\mu - A\sigma \le X \le \mu + A\sigma)$], may be calculated as F(A) − F(–A), where F(A) is the cumulative standard normal probability of *A*, or as 1 − 2[F(–A)].

## LOS 9.l

The standard normal probability distribution has a mean of 0 and a standard deviation of 1.

A normally distributed random variable *X* can be standardized as $Z = \frac{x-\mu}{\sigma}$ and *Z* will be normally distributed with mean = 0 and standard deviation 1.

The *z*-table is used to find the probability that *X* will be less than or equal to a given value.

- $P(X < x) = F(x) = F\left[\frac{x-\mu}{\sigma}\right] = F(z)$, which is found in the standard normal probability table.
- $P(X > x) = 1 - P(X < x) = 1 - F(z)$.

## LOS 9.m

The safety-first ratio for portfolio P, based on a target return $R_T$ , is:

$$\text{SFRatio} = \frac{E(R_P) - R_T}{\sigma_P}$$

Shortfall risk is the probability that a portfolio's value (or return) will fall below a specific value over a given period of time. Greater safety-first ratios are preferred and indicate a smaller shortfall probability. Roy's safety-first criterion states that the optimal portfolio minimizes shortfall risk.

## LOS 9.n

If *x* is normally distributed, $e^x$ follows a lognormal distribution. A lognormal distribution is often used to model asset prices, since a lognormal random variable cannot be negative and can take on any positive value.

## LOS 9.o

As we decrease the length of discrete compounding periods (e.g., from quarterly to monthly) the effective annual rate increases. As the length of the compounding period in discrete compounding gets shorter and shorter, the compounding becomes continuous, where the effective annual rate = $e^i - 1$.

For a holding period return (HPR) over any period, the equivalent continuously compounded rate over the period is ln(1 + HPR).

## LOS 9.p

Monte Carlo simulation uses randomly generated values for risk factors, based on their assumed distributions, to produce a distribution of possible security values. Its limitations are that it is fairly complex and will provide answers that are no better than the assumptions used.

### LOS 9.q

Historical simulation uses randomly selected past changes in risk factors to generate a distribution of possible security values, in contrast to Monte Carlo simulation, which uses randomly generated values. A limitation of historical simulation is that it cannot consider the effects of significant events that did not occur in the sample period.

# ANSWER KEY FOR MODULE QUIZZES

## Module Quiz 9.1

1. **B** Time is usually a continuous random variable; the others are discrete. (LOS 9.a)

2. **B** For a continuous distribution p(x) = 0 for all *X*; only ranges of value of *X* have positive probabilities. (LOS 9.a)

3. **B** $\frac{8-5}{5} + \frac{8-10}{5} = \frac{1}{5}$, and $\frac{8-10}{5}$ is negative, so this satisfies neither of the requirements for a probability distribution. The others have $P[X_i]$ between zero and one and $\sum P[X_i] = 1$, and thus satisfy both requirements for a probability distribution. (LOS 9.a)

4. **B** From the table. (LOS 9.d)

5. **C** (0.04 + 0.11 + 0.18 + 0.24 + 0.14 + 0.17) = 0.88 (LOS 9.c)

6. **B** (0.14 + 0.17 + 0.09 + 0.03) = 0.43 (LOS 9.d)

7. **C** (0.18 + 0.24 + 0.14 + 0.17) = 0.73 (LOS 9.d)

8. **A** 0 + 1(0.11) + 2(0.18) + 3(0.24) + 4(0.14) + 5(0.17) + 6(0.09) + 7(0.03) = 3.35 (LOS 9.d)

9. **A** There may be any number of independent trials, each with only two possible outcomes. (LOS 9.e)

10. **B** With only two possible outcomes, there must be some positive probability for each. If this were not the case, the variable in question would not be a random variable, and a probability distribution would be meaningless. It does not matter if one of the possible outcomes happens to be zero. (LOS 9.e)

11. **C** Success = having a fax machine. [6! / 4!(6 − 4)!](0.6)4(0.4)6 − 4 = 15(0.1296)(0.16) = 0.311. (LOS 9.f)

12. **A** Success = staying for five years. [6! / 2!(6 − 2)!](0.10)2(0.90)6 − 2 = 15(0.01)(0.656) = 0.0984. (LOS 9.f)

13. **C** Success = passing the exam. Then, E(success) = np = 15 × 0.4 = 6. (LOS 9.f)

14. **C** F(x) is the cumulative probability, P(x < 20) here. Because all the observations in this distribution are between 4 and 10, the probability of an outcome less than 20 is 100%. (LOS 9.h)

## Module Quiz 9.2

1. **A** Normal distributions are symmetrical (i.e., have zero skewness) and their kurtosis is equal to three. (LOS 9.i)

2. **B** To describe a multivariate normal distribution, we must consider the correlations among the variables as well as the means and variances of the variables. (LOS 9.j)

3. **C** 1 − F(–1) = F(1) = 0.8413. There is an 84.13% probability that a randomly chosen income is not more than one standard deviation below the mean. (LOS 9.k)

4. **B** This is true by the formula for *z*. (LOS 9.l)

5. **C** By the symmetry of the *z*-distribution and F(0) = 0.5. Half the distribution lies on each side of the mean. (LOS 9.l)

### Module Quiz 9.3

1. **C** SFR = (18 − 4) / 40 = 0.35 is the largest value. (LOS 9.m)

2. **A** SFR = (5 − 0) / 8 = 0.625 is the largest value. (LOS 9.m)

3. **A** Using the tables, the cdf for –1.3 is 9.68%, which is the probability of returns less than 2%. (LOS 9.m)

4. **B** A lognormally distributed variable is never negative. (LOS 9.n)

5. **B** ln(23 / 20) = 0.1398 (LOS 9.o)

6. **B** ln(2) = 0.6931 (LOS 9.o)

7. **B** Monte Carlo simulation involves modeling asset prices or returns by generating random values for the risk factors that affect the price of a security. (LOS 9.p, LOS 9.q)

The following is a review of the Quantitative Methods (2) principles designed to address the learning outcome statements set forth by CFA Institute. Cross-Reference to CFA Institute Assigned Reading #10.

# READING 10: SAMPLING AND ESTIMATION

**Study Session 3**

## EXAM FOCUS

This topic review covers random samples and inferences about population means from sample data. It is essential that you know the central limit theorem, for it allows us to use sampling statistics to construct confidence intervals for point estimates of population means. Make sure you can calculate confidence intervals for population means given sample parameter estimates and a level of significance, and know when it is appropriate to use the *z*-statistic versus the *t*-statistic. You should also understand the basic procedures for creating random samples, and recognize the warning signs of various sampling biases from nonrandom samples.

## MODULE 10.1: CENTRAL LIMIT THEOREM AND STANDARD ERROR

Video covering this content is available online.

In many real-world statistics applications, it is impractical (or impossible) to study an entire population. When this is the case, a subgroup of the population, called a sample, can be evaluated. Based upon this sample, the parameters of the underlying population can be estimated.

For example, rather than attempting to measure the performance of the U.S. stock market by observing the performance of all 10,000 or so stocks trading in the United States at any one time, the performance of the subgroup of 500 stocks in the S&P 500 can be measured. The results of the statistical analysis of this sample can then be used to draw conclusions about the entire population of U.S. stocks.

**LOS 10.a: Define simple random sampling and a sampling distribution.**

**LOS 10.b: Explain sampling error.**

*CFA® Program Curriculum, Volume 1, page 570*

**Simple random sampling** is a method of selecting a sample in such a way that each item or person in the population being studied has the same likelihood of being included in the sample. As an example of simple random sampling, assume that you want to draw a sample of five items out of a group of 50 items. This can be accomplished by numbering each of the 50 items, placing them in a hat, and shaking the hat. Next, one number can be drawn randomly from the hat. Repeating this process (experiment) four more times results in a set of five numbers. The five drawn numbers (items) comprise a simple random sample from the population. In applications like this one, a random-number table or a computer random-

number generator is often used to create the sample. Another way to form an approximately random sample is **systematic sampling**, selecting every *n*th member from a population.

**Sampling error** is the difference between a sample statistic (the mean, variance, or standard deviation of the sample) and its corresponding population parameter (the true mean, variance, or standard deviation of the population). For example, the sampling error for the mean is as follows:

sampling error of the mean = sample mean − population mean = $\bar{x} - \mu$

## A Sampling Distribution

It is important to recognize that the sample statistic itself is a random variable and, therefore, has a probability distribution. The **sampling distribution** of the sample statistic is a probability distribution of all possible sample statistics computed from a set of equal-size samples that were randomly drawn from the same population. Think of it as the probability distribution of a statistic from many samples.

For example, suppose a random sample of 100 bonds is selected from a population of a major municipal bond index consisting of 1,000 bonds, and then the mean return of the 100-bond sample is calculated. Repeating this process many times will result in many different estimates of the population mean return (i.e., one for each sample). The distribution of these estimates of the mean is the *sampling distribution of the mean*.

It is important to note that this sampling distribution is distinct from the distribution of the actual prices of the 1,000 bonds in the underlying population and will have different parameters.

**LOS 10.c: Distinguish between simple random and stratified random sampling.**

*CFA® Program Curriculum, Volume 1, page 572*

**Stratified random sampling** uses a classification system to separate the population into smaller groups based on one or more distinguishing characteristics. From each subgroup, or stratum, a random sample is taken and the results are pooled. The size of the samples from each stratum is based on the size of the stratum relative to the population.

Stratified sampling is often used in bond indexing because of the difficulty and cost of completely replicating the entire population of bonds. In this case, bonds in a population are categorized (stratified) according to major bond risk factors such as duration, maturity, coupon rate, and the like. Then, samples are drawn from each separate category and combined to form a final sample.

To see how this works, suppose you want to construct a bond portfolio that is indexed to the major municipal bond index using a stratified random sampling approach. First, the entire population of 1,000 municipal bonds in the index can be classified on the basis of maturity and coupon rate. Then, cells (stratum) can be created for different maturity/coupon combinations, and random samples can be drawn from each of the maturity/coupon cells. To sample from a cell containing 50 bonds with 2- to 4-year maturities and coupon rates less than 5%, we would select five bonds. The number of bonds drawn from a given cell corresponds to the cell's weight relative to the population (index), or (50 / 1000) × (100) = 5 bonds. This process is repeated for all of the maturity/coupon cells, and the individual samples are combined to form the portfolio.

By using stratified sampling, we guarantee that we sample five bonds from this cell. If we had used simple random sampling, there would be no guarantee that we would sample any of the bonds in the cell. Or, we may have selected more than five bonds from this cell.

### LOS 10.d: Distinguish between time-series and cross-sectional data.

*CFA® Program Curriculum, Volume 1, page 573*

**Time-series data** consist of observations taken *over a period of time* at specific and equally spaced time intervals. The set of monthly returns on Microsoft stock from January 1994 to January 2004 is an example of a time-series data sample.

**Cross-sectional data** are a sample of observations taken *at a single point in time*. The sample of reported earnings per share of all Nasdaq companies as of December 31, 2004, is an example of a cross-sectional data sample.

Time-series and cross-sectional data can be pooled in the same data set. **Longitudinal data** are observations over time of multiple characteristics of the same entity, such as unemployment, inflation, and GDP growth rates for a country over 10 years. **Panel data** contain observations over time of the same characteristic for multiple entities, such as debt/equity ratios for 20 companies over the most recent 24 quarters. Panel and longitudinal data are typically presented in table or spreadsheet form.

### LOS 10.e: Explain the central limit theorem and its importance.

*CFA® Program Curriculum, Volume 1, page 576*

The **central limit theorem** states that for simple random samples of size *n* from a *population* with a mean μ and a finite variance $\sigma^2$, the sampling distribution of the sample mean *x* approaches a normal probability distribution with mean μ and a variance equal to $\frac{\sigma^2}{n}$ as the sample size becomes large.

The central limit theorem is extremely useful because the normal distribution is relatively easy to apply to hypothesis testing and to the construction of confidence intervals. Specific inferences about the population mean can be made from the sample mean, *regardless of the population's distribution*, as long as the sample size is "sufficiently large," which usually means $n \geq 30$.

*Important properties of the central limit theorem* include the following:

- If the sample size *n* is sufficiently large ($n \geq 30$), the sampling distribution of the sample means will be approximately normal. Remember what's going on here, random samples of size *n* are repeatedly being taken from an overall larger population. Each of these random samples has its own mean, which is itself a random variable, and this set of sample means has a distribution that is approximately normal.
- The mean of the population, μ, and the mean of the distribution of all possible sample means are equal.
- The variance of the distribution of sample means is $\frac{\sigma^2}{n}$, the population variance divided by the sample size.

### LOS 10.f: Calculate and interpret the standard error of the sample mean.

The **standard error of the sample mean** is the standard deviation of the distribution of the sample means.

When the standard deviation of the population, σ, is *known*, the standard error of the sample mean is calculated as:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

where:
$\sigma_{\bar{x}}$ = standard error of the sample mean
σ = standard deviation of the population
n = size of the sample

**EXAMPLE: Standard error of sample mean (known population variance)**

The mean hourly wage for Iowa farm workers is $13.50 with a *population standard deviation* of $2.90. Calculate and interpret the standard error of the sample mean for a sample size of 30.

**Answer:**

Because the population standard deviation, σ, *is known*, the standard error of the sample mean is expressed as:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{\$2.90}{\sqrt{30}} = \$0.53$$

This means that if we were to take many samples of size 30 from the Iowa farm worker population and prepare a sampling distribution of the sample means, we would get a distribution with an expected mean of $13.50 and standard error (standard deviation of the sample means) of $0.53.

Practically speaking, the *population's standard deviation is almost never known*. Instead, the standard error of the sample mean must be estimated by dividing the standard deviation of *the sample* by $\sqrt{n}$ :

$$s_{\bar{x}} = \frac{s}{\sqrt{n}}$$

**EXAMPLE: Standard error of sample mean (unknown population variance)**

Suppose a sample contains the past 30 monthly returns for McCreary, Inc. The mean return is 2% and the *sample* standard deviation is 20%. Calculate and interpret the standard error of the sample mean.

**Answer:**

Since σ is unknown, the standard error of the sample mean is:

$$s_{\bar{x}} = \frac{s}{\sqrt{n}} = \frac{20\%}{\sqrt{30}} = 3.6\%$$

This implies that if we took all possible samples of size 30 from McCreary's monthly returns and prepared a sampling distribution of the sample means, the mean would be 2% with a standard error of 3.6%.

**EXAMPLE: Standard error of sample mean (unknown population variance)**

Continuing with our example, suppose that instead of a sample size of 30, we take a sample of the past 200 monthly returns for McCreary, Inc. In order to highlight the effect of sample size on the sample standard error, let's assume that the mean return and standard deviation of this larger sample remain at 2% and 20%, respectively. Now, calculate the standard error of the sample mean for the 200-return sample.

**Answer:**

The standard error of the sample mean is computed as:

$$s_{\bar{x}} = \frac{s}{\sqrt{n}} = \frac{20\%}{\sqrt{200}} = 1.4\%$$

The result of the preceding two examples illustrates an important property of sampling distributions. Notice that the value of the standard error of the sample mean decreased from 3.6% to 1.4% as the sample size increased from 30 to 200. This is because as the sample size increases, the sample mean gets closer, on average, to the true mean of the population. In other words, the distribution of the sample means about the population mean gets smaller and smaller, so the standard error of the sample mean decreases.

### PROFESSOR'S NOTE

I get a lot of questions about when to use σ and $\sigma/\sqrt{n}$. Just remember that the standard deviation of the means of multiple samples is less than the standard deviation of single observations. If the standard deviation of monthly stock returns is 2%, the standard error (deviation) of the average monthly return over the next six months is 2% / $\sqrt{6}$ = 0.82%. The average of several observations of a random variable will be less widely dispersed (have lower standard deviation) around the expected value than will a single observation of the random variable.

## LOS 10.g: Identify and describe desirable properties of an estimator.

*CFA® Program Curriculum, Volume 1, page 580*

Regardless of whether we are concerned with point estimates or confidence intervals, there are certain statistical properties that make some estimates more desirable than others. These **desirable properties of an estimator are unbiasedness, efficiency**, and **consistency**.

- An *unbiased* estimator is one for which the expected value of the estimator is equal to the parameter you are trying to estimate. For example, because the expected value of the sample mean is equal to the population mean $[E(\bar{x}) = \mu]$, the sample mean is an unbiased estimator of the population mean.
- An unbiased estimator is also *efficient* if the variance of its sampling distribution is smaller than all the other unbiased estimators of the parameter you are trying to estimate. The sample mean, for example, is an unbiased and efficient estimator of the population mean.
- A *consistent* estimator is one for which the accuracy of the parameter estimate increases as the sample size increases. As the sample size increases, the standard error of the sample mean falls, and the sampling distribution bunches more closely around the population mean. In fact, as the sample size approaches infinity, the standard error approaches zero.

### MODULE QUIZ 10.1

*To best evaluate your performance, enter your quiz answers online.*

1. A simple random sample is a sample drawn in such a way that each member of the population has:
   A. some chance of being selected in the sample.
   B. an equal chance of being included in the sample.
   C. a 1% chance of being included in the sample.
2. The sampling distribution of a statistic is the probability distribution made up of all possible:

A. observations from the underlying population.
B. sample statistics computed from samples of varying sizes drawn from the same population.
C. sample statistics computed from samples of the same size drawn from the same population.

3. Sampling error is defined as:
   A. an error that occurs when a sample of less than 30 elements is drawn.
   B. an error that occurs during collection, recording, and tabulation of data.
   C. the difference between the value of a sample statistic and the value of the corresponding population parameter.
4. The mean age of all CFA candidates is 28 years. The mean age of a random sample of 100 candidates is found to be 26.5 years. The difference of 1.5 years is called:
   A. the random error.
   B. the sampling error.
   C. the population error.
5. The sample of debt/equity ratios of 25 publicly traded U.S. banks as of fiscal year-end 2003 is an example of:
   A. a point estimate.
   B. cross-sectional data.
   C. a stratified random sample.
6. To apply the central limit theorem to the sampling distribution of the sample mean, the sample is usually considered to be large if *n* is *greater* than:
   A. 20.
   B. 25.
   C. 30.
7. If *n* is large and the population standard deviation is unknown, the standard error of the sampling distribution of the sample mean is *equal* to:
   A. the sample standard deviation divided by the sample size.
   B. the population standard deviation multiplied by the sample size.
   C. the sample standard deviation divided by the square root of the sample size.
8. The standard error of the sampling distribution of the sample mean for a sample size of *n* drawn from a population with a mean of μ and a standard deviation of σ is:
   A. sample standard deviation divided by the sample size.
   B. sample standard deviation divided by the square root of the sample size.
   C. population standard deviation divided by the square root of the sample size.
9. Assume that a population has a mean of 14 with a standard deviation of 2. If a random sample of 49 observations is drawn from this population, the standard error of the sample mean is *closest* to:
   A. 0.04.
   B. 0.29.
   C. 2.00.
10. The population's mean is 30 and the mean of a sample of size 100 is 28.5. The variance of the sample is 25. The standard error of the sample mean is *closest* to:
    A. 0.05.
    B. 0.25.
    C. 0.50.
11. Which of the following is *least likely* a desirable property of an estimator?
    A. Reliability.
    B. Efficiency.
    C. Consistency.

## MODULE 10.2: CONFIDENCE INTERVALS AND *t*-DISTRIBUTION

Video covering this content is available online.

### LOS 10.h: Distinguish between a point estimate and a confidence interval estimate of a population parameter.

*CFA® Program Curriculum, Volume 1, page 579*

**Point estimates** are single (sample) values used to estimate population parameters. The formula used to compute the point estimate is called the estimator. For example, the sample mean, *x*, is an estimator of the population mean μ and is computed using the familiar formula:

$$\bar{x} = \frac{\sum x}{n}$$

The value generated with this calculation for a given sample is called the *point estimate* of the mean.

A *confidence interval* is a range of values in which the population parameter is expected to lie. The construction of confidence intervals is described later in this topic review.

### LOS 10.i: Describe properties of Student's *t*-distribution and calculate and interpret its degrees of freedom.

*CFA® Program Curriculum, Volume 1, page 584*

**Student's *t*-distribution**, or simply the *t*-distribution, is a bell-shaped probability distribution that is symmetrical about its mean. It is the appropriate distribution to use when constructing confidence intervals based on *small samples* (n < 30) from populations with *unknown variance* and a normal, or approximately normal, distribution. It may also be appropriate to use the *t*-distribution when the population variance is unknown and the sample size is large enough that the central limit theorem will assure that the sampling distribution is approximately normal.

Student's *t*-distribution has the following properties:

- It is symmetrical.
- It is defined by a single parameter, the **degrees of freedom** (df), where the degrees of freedom are equal to the number of sample observations minus 1, n − 1, for sample means.
- It has more probability in the tails ("fatter tails") than the normal distribution.
- As the degrees of freedom (the sample size) gets larger, the shape of the *t*-distribution more closely approaches a standard normal distribution.

When *compared to the normal distribution*, the *t*-distribution is flatter with more area under the tails (i.e., it has fatter tails). As the degrees of freedom for the *t*-distribution increase, however, its shape approaches that of the normal distribution.

The degrees of freedom for tests based on sample means are n − 1 because, given the mean, only n − 1 observations can be unique.

The *t*-distribution is a symmetrical distribution that is centered about zero. The shape of the *t*-distribution is dependent on the number of degrees of freedom, and degrees of freedom are based on the number of sample observations. The *t*-distribution is flatter and has thicker tails than the standard normal distribution. As the number of observations increases (i.e., the degrees of freedom increase), the *t*-distribution becomes more spiked and its tails become thinner. As the number of degrees of freedom increases without bound, the *t*-distribution

converges to the standard normal distribution (*z*-distribution). The thickness of the tails relative to those of the *z*-distribution is important in hypothesis testing because thicker tails mean more observations away from the center of the distribution (more outliers). Hence, hypothesis testing using the *t*-distribution makes it more difficult to reject the null relative to hypothesis testing using the *z*-distribution.

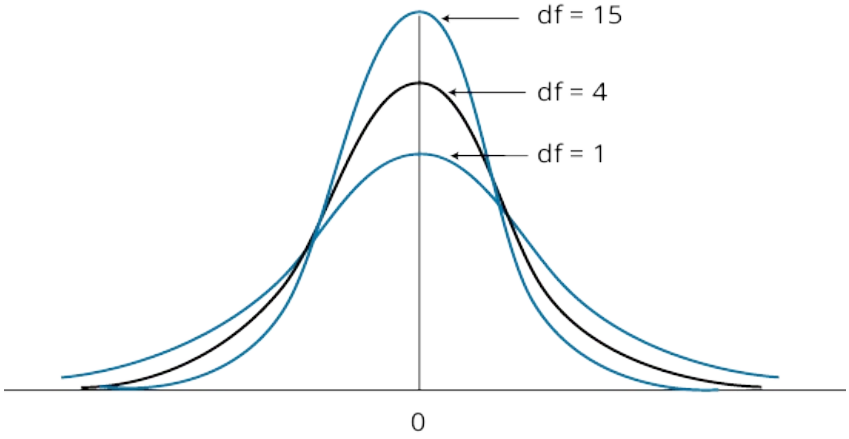
The table in Figure 10.1 contains one-tailed critical values for the *t*-distribution at the 0.05 and 0.025 levels of significance with various degrees of freedom (df). Note that, unlike the *z*-table, the *t*-values are contained within the table, and the probabilities are located at the column headings. Also note that the level of significance of a *t*-test corresponds to the *one-tailed probabilities*, *p*, that head the columns in the *t*-table.

**Figure 10.1: Table of Critical *t*-Values**

| | **One-Tailed Probabilities, p** | |
|---|---|---|
| df | p = 0.05 | p = 0.025 |
| 5 | 2.015 | 2.571 |
| 10 | 1.812 | 2.228 |
| 15 | 1.753 | 2.131 |
| 20 | 1.725 | 2.086 |
| 25 | 1.708 | 2.060 |
| 30 | 1.697 | 2.042 |
| 40 | 1.684 | 2.021 |
| 50 | 1.676 | 2.009 |
| 60 | 1.671 | 2.000 |
| 70 | 1.667 | 1.994 |
| 80 | 1.664 | 1.990 |
| 90 | 1.662 | 1.987 |
| 100 | 1.660 | 1.984 |
| 120 | 1.658 | 1.980 |
| ∞ | 1.645 | 1.960 |

Figure 10.2 illustrates the different shapes of the *t*-distribution associated with different degrees of freedom. The tendency is for the *t*-distribution to look more and more like the normal distribution as the degrees of freedom increase. Practically speaking, the greater the degrees of freedom, the greater the percentage of observations near the center of the distribution and the lower the percentage of observations in the tails, which are thinner as degrees of freedom increase. This means that confidence intervals for a random variable that follows a *t*-distribution must be wider (narrower) when the degrees of freedom are less (more) for a given significance level.

**Figure 10.2: *t*-Distributions for Different Degrees of Freedom (df)**




### LOS 10.j: Calculate and interpret a confidence interval for a population mean, given a normal distribution with 1) a known population variance, 2) an unknown population variance, or 3) an unknown population variance and a large sample size.

*CFA® Program Curriculum, Volume 1, page 581*

**Confidence interval** estimates result in a range of values within which the actual value of a parameter will lie, given the probability of 1 − α. Here, alpha, α, is called the *level of significance* for the confidence interval, and the probability 1 − α is referred to as the *degree of confidence*. For example, we might estimate that the population mean of random variables will range from 15 to 25 with a 95% degree of confidence, or at the 5% level of significance.

Confidence intervals are usually constructed by adding or subtracting an appropriate value from the point estimate. In general, confidence intervals take on the following form:

point estimate ± (reliability factor × standard error)

where:
point estimate = value of a sample statistic of the population parameter
reliability factor = number that depends on the sampling distribution of the point estimate and the probability that the point estimate falls in the confidence interval, (1 − α)
standard error = standard error of the point estimate

If the population has a *normal distribution with a known variance*, a **confidence interval for the population mean** can be calculated as:

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

where:
$\bar{x}$ = *point estimate* of the population mean (sample mean).
$z_{\alpha/2}$ = *reliability factor*, a standard normal random variable for which the probability in the right-hand tail of the distribution is α/2. In other words, this is the *z*-score that leaves α/2 of probability in the upper tail.
$\frac{\sigma}{\sqrt{n}}$ = the *standard error* of the sample mean where σ is the known standard deviation of the population, and *n* is the sample size.

The most commonly used standard normal distribution reliability factors are:

$z_{\alpha/2}$ = 1.645 for 90% confidence intervals (the significance level is 10%, 5% in each tail).

$z_{\alpha/2}$ = 1.960 for 95% confidence intervals (the significance level is 5%, 2.5% in each tail).

$z_{\alpha/2}$ = 2.575 for 99% confidence intervals (the significance level is 1%, 0.5% in each tail).

Do these numbers look familiar? They should! In our review of common probability distributions, we found the probability under the standard normal curve between z = –1.96 and z = +1.96 to be 0.95, or 95%. Owing to symmetry, this leaves a probability of 0.025 under each tail of the curve beyond z = –1.96 or z = +1.96, for a total of 0.05, or 5%—just what we need for a significance level of 0.05, or 5%.

**EXAMPLE: Confidence interval**

Consider a practice exam that was administered to 36 Level I candidates. The mean score on this practice exam was 80. Assuming a population standard deviation equal to 15, construct and interpret a 99% confidence interval for the mean score on the practice exam for 36 candidates. *Note that in this example the population standard deviation is known, so we don't have to estimate it*.

**Answer:**

At a confidence level of 99%, $z_{\alpha/2} = z_{0.005}$ = 2.58. So, the 99% confidence interval is calculated as follows:

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 80 \pm 2.58\frac{15}{\sqrt{36}} = 80 \pm 6.45$$

Thus, the 99% confidence interval ranges from 73.55 to 86.45.

**EXAMPLE: Confidence intervals for a population mean and for a single observation**

Annual returns on energy stocks are approximately normally distributed with a mean of 9% and standard deviation of 6%. Construct a 90% confidence interval for the annual returns of a randomly selected energy stock and a 90% confidence interval for the mean of the annual returns for a sample of 12 energy stocks.

**Answer:**

A 90% confidence interval for a single observation is 1.645 *standard deviations* from the sample mean.

9% ± 1.645(6%) = –0.87% to 18.87%

A 90% confidence interval for the population mean is 1.645 *standard errors* from the sample mean.

$$9\% \pm 1.645\left(\frac{6\%}{\sqrt{12}}\right) = 6.15\% \text{ to } 11.85\%$$

Confidence intervals can be interpreted from a probabilistic perspective or a practical perspective. With regard to the outcome of the CFA® practice exam example, these two perspectives can be described as follows:

- *Probabilistic interpretation*. After repeatedly taking samples of CFA candidates, administering the practice exam, and constructing confidence intervals for each sample's mean, 99% of the resulting confidence intervals will, in the long run, include the population mean.
- *Practical interpretation*. We are 99% confident that the population mean score is between 73.55 and 86.45 for candidates from this population.

## Confidence Intervals for the Population Mean: Normal With Unknown Variance

If the distribution of the *population is normal with unknown variance*, we can use the *t*-distribution to construct a confidence interval:

$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$$

where:

$\bar{x}$ = the point estimate of the population mean

$t_{\alpha/2}$ = the *t*-reliability factor (a.k.a. *t*-statistic or critical *t*-value) corresponding to a *t*-distributed random variable with n – 1 degrees of freedom, where *n* is the sample size. The area under the tail of the *t*-distribution to the right of $t_{\alpha/2}$ is α/2

$\frac{s}{\sqrt{n}}$ = standard error of the sample mean

s = sample standard deviation

Unlike the standard normal distribution, the reliability factors for the *t*-distribution depend on the sample size, so we can't rely on a commonly used set of reliability factors. Instead, reliability factors for the *t*-distribution have to be looked up in a table of Student's *t*-distribution, like the one at the back of this book.

Owing to the relatively fatter tails of the *t*-distribution, confidence intervals constructed using *t*-reliability factors ($t_{\alpha/2}$) will be more conservative (wider) than those constructed using *z*-reliability factors ($z_{\alpha/2}$).

**EXAMPLE: Confidence intervals**

Let's return to the McCreary, Inc. example. Recall that we took a sample of the past 30 monthly stock returns for McCreary, Inc. and determined that the mean return was 2% and the sample standard deviation was 20%. Since the population variance is unknown, the standard error of the sample was estimated to be:

$$s_{\bar{x}} = \frac{s}{\sqrt{n}} = \frac{20\%}{\sqrt{30}} = 3.6\%$$

Now, let's construct a 95% confidence interval for the mean monthly return.

**Answer:**

Here, we will use the *t*-reliability factor because the population variance is unknown. Since there are 30 observations, the degrees of freedom are 29 = 30 – 1. Remember, because this is a two-tailed test at the 95% confidence level, the probability under each tail must be α/2 = 2.5%, for a total of 5%. So, referencing the one-tailed probabilities for Student's *t*-distribution at the back of this book, we find the critical *t*-value (reliability factor) for α/2 = 0.025 and df = 29 to be $t_{29,\ 2.5}$= 2.045. Thus, the 95% confidence interval for the population mean is:

$$2\% \pm 2.045\left(\frac{20\%}{\sqrt{30}}\right) = 2\% \pm 2.045\,(3.6\%) = 2\% \pm 7.4\%$$

Thus, the 95% confidence has a lower limit of –5.4% and an upper limit of +9.4%.

We can interpret this confidence interval by saying that we are 95% confident that the population mean monthly return for McCreary stock is between –5.4% and +9.4%.

### PROFESSOR'S NOTE

You should practice looking up reliability factors (a.k.a. critical *t*-values or *t*-statistics) in a *t*-table. The first step is always to compute the degrees of freedom, which is n – 1. The second step is to find the appropriate level of alpha or significance. This depends on whether the test you're concerned with is one-tailed (use α) or two-tailed (use α/2). In this

review, our tests will always be two-tailed because confidence intervals are designed to compute an upper and lower limit. Thus, we will use α/2. To look up $t_{29,\ 2.5}$, find the 29 df row and match it with the 0.025 column; t = 2.045 is the result. We'll do more of this in our study of hypothesis testing.

## Confidence Interval for a Population Mean When the Population Variance Is Unknown Given a Large Sample From Any Type of Distribution

We now know that the *z*-statistic should be used to construct confidence intervals when the population distribution is normal and the variance is known, and the *t*-statistic should be used when the distribution is normal but the variance is unknown. But what do we do when the distribution is *nonnormal*?

As it turns out, the size of the sample influences whether or not we can construct the appropriate confidence interval for the sample mean.

- If the *distribution is nonnormal* but the *population variance is known*, the z-statistic can be used as long as the sample size is large (n ≥ 30). We can do this because the central limit theorem assures us that the distribution of the sample mean is approximately normal when the sample is large.
- If the *distribution is nonnormal* and the *population variance is unknown*, the *t*-statistic can be used as long as the sample size is large (n ≥ 30). It is also acceptable to use the *z*-statistic, although use of the *t*-statistic is more conservative.

This means that if we are sampling from a nonnormal distribution (which is sometimes the case in finance), *we cannot create a confidence interval if the sample size is less than 30*. So, all else equal, make sure you have a sample of at least 30, and the larger, the better.

Figure 10.3 summarizes this discussion.

### PROFESSOR'S NOTE

You should commit the criteria in the following table to memory.

**Figure 10.3: Criteria for Selecting the Appropriate Test Statistic**

| When sampling from a: | Test Statistic | |
|---|---|---|
| | **Small Sample (n < 30)** | **Large Sample (n ≥ 30)** |
| *Normal* distribution with *known* variance | *z*-statistic | *z*-statistic |
| *Normal* distribution with *unknown* variance | *t*-statistic | *t*-statistic* |
| *Nonnormal* distribution with *known* variance | not available | *z*-statistic |
| *Nonnormal* distribution with *unknown* variance | not available | *t*-statistic* |

*The *z*-statistic is theoretically acceptable here, but use of the *t*-statistic is more conservative.

All of the preceding analysis depends on the sample we draw from the population being random. If the sample isn't random, the central limit theorem doesn't apply, our estimates won't have the desirable properties, and we can't form unbiased confidence intervals. Surprisingly, creating a *random sample* is not as easy as one might believe. There are a number of potential mistakes in sampling methods that can bias the results. These biases are

particularly problematic in financial research, where available historical data are plentiful, but the creation of new sample data by experimentation is restricted.

**LOS 10.k: Describe the issues regarding selection of the appropriate sample size, data-mining bias, sample selection bias, survivorship bias, look-ahead bias, and time-period bias.**

*CFA® Program Curriculum, Volume 1, page 589*

We have seen so far that a larger sample reduces the sampling error and the standard deviation of the sample statistic around its true (population) value. Confidence intervals are narrower when samples are larger and the standard errors of the point estimates of population parameters are less.

There are two limitations on this idea of "larger is better" when it comes to selecting an appropriate sample size. One is that larger samples may contain observations from a different population (distribution). If we include observations which come from a different population (one with a different population parameter), we will not necessarily improve, and may even reduce, the precision of our population parameter estimates. The other consideration is cost. The costs of using a larger sample must be weighed against the value of the increase in precision from the increase in sample size. Both of these factors suggest that the largest possible sample size is not always the most appropriate choice.

## Data-Mining Bias, Sample Selection Bias, Survivorship Bias, Look-Ahead Bias, and Time-Period Bias

**Data mining** occurs when analysts repeatedly use the same database to search for patterns or trading rules until one that "works" is discovered. For example, empirical research has provided evidence that value stocks appear to outperform growth stocks. Some researchers argue that this anomaly is actually the product of data mining. Because the data set of historical stock returns is quite limited, it is difficult to know for sure whether the difference between value and growth stock returns is a true economic phenomenon, or simply a chance pattern that was stumbled upon after repeatedly looking for any identifiable pattern in the data.

**Data-mining bias** refers to results where the statistical significance of the pattern is overestimated because the results were found through data mining.

When reading research findings that suggest a profitable trading strategy, make sure you heed the following warning signs of data mining:

- Evidence that many different variables were tested, most of which are unreported, until significant ones were found.
- The lack of any economic theory that is consistent with the empirical results.

The best way to avoid data mining is to test a potentially profitable trading rule on a data set different from the one you used to develop the rule (i.e., use out-of-sample data).

**Sample selection bias** occurs when some data is systematically excluded from the analysis, usually because of the lack of availability. This practice renders the observed sample to be nonrandom, and any conclusions drawn from this sample can't be applied to the population

because the observed sample and the portion of the population that was not observed are different.

**Survivorship bias** is the most common form of sample selection bias. A good example of the existence of survivorship bias in investments is the study of mutual fund performance. Most mutual fund databases, like Morningstar®'s, only include funds currently in existence—the "survivors." They do not include funds that have ceased to exist due to closure or merger.

This would not be a problem if the characteristics of the surviving funds and the missing funds were the same; then the sample of survivor funds would still be a random sample drawn from the population of mutual funds. As one would expect, however, and as evidence has shown, the funds that are dropped from the sample have lower returns relative to the surviving funds. Thus, the surviving sample is biased toward the better funds (i.e., it is not random). The analysis of a mutual fund sample with survivorship bias will yield results that overestimate the average mutual fund return because the database only includes the better-performing funds. The solution to survivorship bias is to use a sample of funds that all started at the same time and not drop funds that have been dropped from the sample.

**Look-ahead bias** occurs when a study tests a relationship using sample data that was not available on the test date. For example, consider the test of a trading rule that is based on the price-to-book ratio at the end of the fiscal year. Stock prices are available for all companies at the same point in time, while end-of-year book values may not be available until 30 to 60 days after the fiscal year ends. In order to account for this bias, a study that uses price-to-book value ratios to test trading strategies might estimate the book value as reported at fiscal year end and the market value two months later.

**Time-period bias** can result if the time period over which the data is gathered is either too short or too long. If the time period is too short, research results may reflect phenomena specific to that time period, or perhaps even data mining. If the time period is too long, the fundamental economic relationships that underlie the results may have changed.

For example, research findings may indicate that small stocks outperformed large stocks during 1980–1985. This may well be the result of time-period bias—in this case, using too short a time period. It's not clear whether this relationship will continue in the future or if it is just an isolated occurrence.

On the other hand, a study that quantifies the relationship between inflation and unemployment during the period from 1940–2000 will also result in time-period bias—because this period is too long, and it covers a fundamental change in the relationship between inflation and unemployment that occurred in the 1980s. In this case, the data should be divided into two subsamples that span the period before and after the change.

## MODULE QUIZ 10.2

*To best evaluate your performance, enter your quiz answers online.*

1. Which of the following is *least likely* a property of Student's *t*-distribution?
   A. As the degrees of freedom get larger, the variance approaches zero.
   B. It is defined by a single parameter, the degrees of freedom, which is equal to n – 1.
   C. It has more probability in the tails and less at the peak than a standard normal distribution.
2. A random sample of 100 computer store customers spent an average of $75 at the store. Assuming the distribution is normal and the population standard deviation is $20, the 95%

confidence interval for the population mean is *closest* to:

A. $71.08 to $78.92.
B. $73.89 to $80.11.
C. $74.56 to $79.44.

3. Best Computers, Inc., sells computers and computer parts by mail. A sample of 25 recent orders showed the mean time taken to ship out these orders was 70 hours with a sample standard deviation of 14 hours. Assuming the population is normally distributed, the 99% confidence interval for the population mean is:
   A. 70 ± 2.80 hours.
   B. 70 ± 6.98 hours.
   C. 70 ± 7.83 hours.
4. What is the *most appropriate* test statistic for constructing confidence intervals for the population mean when the population is normally distributed, but the variance is unknown?
   A. The *z*-statistic at α with *n* degrees of freedom.
   B. The *t*-statistic at α/2 with *n* degrees of freedom.
   C. The *t*-statistic at α/2 with n – 1 degrees of freedom.
5. When constructing a confidence interval for the population mean of a nonnormal distribution when the population variance is unknown and the sample size is large (n > 30), an analyst may acceptably use:
   A. either a *z*-statistic or a *t*-statistic.
   B. only a *z*-statistic at α with *n* degrees of freedom.
   C. only a *t*-statistic at α/2 with *n* degrees of freedom.
6. Jenny Fox evaluates managers who have a cross-sectional population standard deviation of returns of 8%. If returns are independent across managers, how large of a sample does Fox need so the standard error of sample means is 1.265%?
   A. 7.
   B. 30.
   C. 40.
7. Annual returns on small stocks have a population mean of 12% and a population standard deviation of 20%. If the returns are normally distributed, a 90% confidence interval on mean returns over a 5-year period is:
   A. 5.40% to 18.60%.
   B. –2.75% to 26.75%.
   C. –5.52% to 29.52%.
8. An analyst who uses historical data that was not publicly available at the time period being studied will have a sample with:
   A. look-ahead bias.
   B. time-period bias.
   C. sample selection bias.
9. Which of the following is *most closely* associated with survivorship bias?
   A. Price-to-book studies.
   B. Stratified bond sampling studies.
   C. Mutual fund performance studies.

## KEY CONCEPTS

### LOS 10.a

Simple random sampling is a method of selecting a sample in such a way that each item or person in the population being studied has the same probability of being included in the sample.

A sampling distribution is the distribution of all values that a sample statistic can take on when computed from samples of identical size randomly drawn from the same population.

### LOS 10.b

Sampling error is the difference between a sample statistic and its corresponding population parameter (e.g., the sample mean minus the population mean).

### LOS 10.c

Stratified random sampling involves randomly selecting samples proportionally from subgroups that are formed based on one or more distinguishing characteristics, so that the sample will have the same distribution of these characteristics as the overall population.

### LOS 10.d

Time-series data consists of observations taken at specific and equally spaced points in time.

Cross-sectional data consists of observations taken at a single point in time.

### LOS 10.e

The central limit theorem states that for a population with a mean μ and a finite variance $\sigma^2$, the sampling distribution of the sample mean of all possible samples of size *n* (for $n \geq 30$) will be approximately normally distributed with a mean equal to μ and a variance equal to $\sigma^2 / n$.

### LOS 10.f

The standard error of the sample mean is the standard deviation of the distribution of the sample means and is calculated as $\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$, where σ, the population standard deviation, is known, and as $s_{\overline{x}} = \frac{s}{\sqrt{n}}$, where s, the sample standard deviation, is used because the population standard deviation is unknown.

### LOS 10.g

Desirable statistical properties of an estimator include unbiasedness (sign of estimation error is random), efficiency (lower sampling error than any other unbiased estimator), and consistency (variance of sampling error decreases with larger sample size).

### LOS 10.h

Point estimates are single value estimates of population parameters. An estimator is a formula used to compute a point estimate.

Confidence intervals are ranges of values, within which the actual value of the parameter will lie with a given probability.

confidence interval = point estimate ± (reliability factor × standard error)

The reliability factor is a number that depends on the sampling distribution of the point estimate and the probability that the point estimate falls in the confidence interval.

### LOS 10.i

The *t*-distribution is similar, but not identical, to the normal distribution in shape—it is defined by the degrees of freedom and has fatter tails compared to the normal distribution.

Degrees of freedom for the *t*-distribution are equal to n − 1. Student's *t*-distribution is closer to the normal distribution when df is greater, and confidence intervals are narrower when df is greater.

### LOS 10.j

For a normally distributed population, a confidence interval for its mean can be constructed using a *z*-statistic when variance is known, and a *t*-statistic when the variance is unknown. The *z*-statistic is acceptable in the case of a normal population with an unknown variance if the sample size is large (30+).

In general, we have:

- $\overline{X} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ when the variance is known, and
- $\overline{X} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$ when the variance is unknown and the sample standard deviation must be used.

### LOS 10.k

Increasing the sample size will generally improve parameter estimates and narrow confidence intervals. The cost of more data must be weighed against these benefits, and adding data that is not generated by the same distribution will not necessarily improve accuracy or narrow confidence intervals.

Potential mistakes in the sampling method can bias results. These biases include data mining (significant relationships that have occurred by chance), sample selection bias (selection is nonrandom), look-ahead bias (basing the test at a point in time on data not available at that time), survivorship bias (using only surviving mutual funds, hedge funds, etc.), and time-period bias (the relation does not hold over other time periods).

# ANSWER KEY FOR MODULE QUIZZES

## Module Quiz 10.1

1. **B** In a simple random sample, each element of the population has an equal probability of being selected. Choice C allows for an equal chance, but only if there are 100 elements in the population from which the random sample is drawn. (LOS 10.a)

2. **C** Suppose you have a population of 10,000 employees. If you take 100 samples of 50 employees each, the distribution of the 100 sample means is the sampling distribution. (LOS 10.a)

3. **C** An example might be the difference between a particular sample mean and the average value of the overall population. (LOS 10.b)

4. **B** The sampling error is the difference between the population parameter and the sample statistic. (LOS 10.b)

5. **B** Cross-sectional data is a set of data that are all collected as of the same point in time. (LOS 10.c, LOS 10.d)

6. **C** Sample sizes of 30 or greater are typically considered large. (LOS 10.e)

7. **C** The formula for the standard error when the population standard deviation is unknown is $s_{\bar{x}} = \frac{s}{\sqrt{n}}$. (LOS 10.f)

8. **C** The formula for the standard error when the population standard deviation is known is $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$. (LOS 10.f)

9. **B** $s_{\overline{X}} = \frac{s}{\sqrt{n}}$. Given $s = 2, s_{\overline{X}} = \frac{2}{\sqrt{49}} = \frac{2}{7} = 0.2857$. (LOS 10.f)

10. **C** $s_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$. Given $\sigma^2 = 25$, $s_{\overline{X}} = \frac{5}{\sqrt{100}} = \frac{5}{10} = 0.5$. (LOS 10.f)

11. **A** Efficiency, consistency, and unbiasedness are desirable properties of an estimator. (LOS 10.g)

## Module Quiz 10.2

1. **A** As the degrees of freedom get larger, the *t*-distribution approaches the normal distribution. As the degrees of freedom fall, the peak of the *t*-distribution flattens and its tails get fatter (more probability in the tails—that's why, all else the same, the critical *t* increases as the *df* decreases). (LOS 10.i)

2. **A** Since the population variance is known and n ≥ 30, the confidence interval is determined as $\bar{x} \pm z_{\alpha/2}(\sigma/\sqrt{n})$. $z_{\alpha/2} = z_{0.025} = 1.96$. So, the confidence interval is 75 ± 1.96(20/10) = 75 ± 3.92 = 71.08 to 78.92. (LOS 10.j)

3. **C** Since the population variance is unknown and n < 30, the confidence interval is determined as $\bar{x} \pm t_{\alpha/2}(s/\sqrt{n})$. Look up $t_{\alpha/2}$ and df = n − 1 to get critical *t*-value.

$t_{0.01/2}$ and df = 24 is 2.797. So, the confidence interval is 70 ± 2.797(14 / 5) = 70 ± 7.83. (LOS 10.j)

4. **C** Use the *t*-statistic at α/2 and n – 1 degrees of freedom when the population variance is unknown. While the *z*-statistic is acceptable when the sample size is large, sample size is not given here, and the *t*-statistic is always appropriate under these conditions. (LOS 10.j)

5. **A** When the sample size is large, and the central limit theorem can be relied upon to assure a sampling distribution that is normal, either the *t*-statistic or the *z*-statistic is acceptable for constructing confidence intervals for the population mean. The *t*-statistic, however, will provide a more conservative range (wider) at a given level of significance. (LOS 10.j)

6. **C** $1.265 = \frac{8}{\sqrt{N}}$, $N = \left(\frac{8}{1.265}\right)^2 \approx 40$ (LOS 10.j)

7. **B** With a known population standard deviation of returns and a normally distributed population, we can use the *z*-distribution. The sample mean for a sample of five years will have a standard deviation of $\frac{20}{\sqrt{5}} = 8.94\%$. A 90% confidence interval around the mean return of 12% is 12% ± 1.65(8.94%) = –2.75% to 26.75%. (LOS 10.j)

8. **A** The primary example of look-ahead bias is using year-end financial information in conjunction with market pricing data to compute ratios like the price/earnings (P/E). The E in the denominator is typically not available for 30–60 days after the end of the period. Hence, data that was available on the test date (P) is mixed with information that was not available (E). That is, the P is “ahead” of the E. (LOS 10.k)

9. **C** Mutual fund performance studies are most closely associated with survivorship bias because only the better-performing funds remain in the sample over time. (LOS 10.k)

The following is a review of the Quantitative Methods (2) designed to address the learning outcome statements set forth by CFA Institute. Cross-Reference to CFA Institute Assigned Reading #11.

# READING 11: HYPOTHESIS TESTING

**Study Session 3**

## EXAM FOCUS

This review addresses common hypothesis testing procedures. These procedures are used to conduct tests of population means, population variances, differences in means, differences in variances, and mean differences. Specific tests reviewed include the *z*-test, *t*-test, chi-square test, and *F*-test. You should know when and how to apply each of these. A standard hypothesis testing procedure is utilized in this review. Know it! You should be able to perform a hypothesis test on the value of the mean without being given any formulas. Confidence intervals, levels of significance, the power of a test, and types of hypothesis testing errors are also discussed. Don't worry about memorizing the messy formulas on testing for the equalities and differences in means and variances at the end of this review, but be able to interpret these statistics.

## MODULE 11.1: HYPOTHESIS TESTS AND TYPES OF ERRORS

Video covering this content is available online.

Hypothesis testing is the statistical assessment of a statement or idea regarding a population. For instance, a statement could be as follows: "The mean return for the U.S. equity market is greater than zero." Given the relevant returns data, hypothesis testing procedures can be employed to test the validity of this statement at a given significance level.

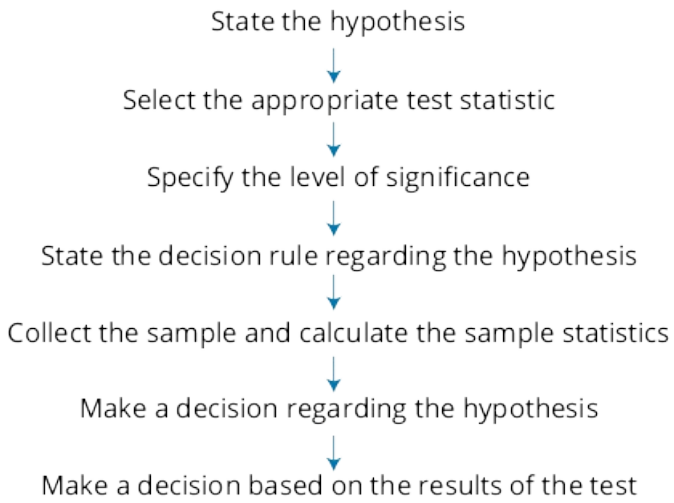
**LOS 11.a: Define a hypothesis, describe the steps of hypothesis testing, and describe and interpret the choice of the null and alternative hypotheses.**

*CFA® Program Curriculum, Volume 1, page 611*

A hypothesis is a statement about the value of a population parameter developed for the purpose of testing a theory or belief. Hypotheses are stated in terms of the population parameter to be tested, like the population mean, μ. For example, a researcher may be interested in the mean daily return on stock options. Hence, the hypothesis may be that the mean daily return on a portfolio of stock options is positive.

Hypothesis testing procedures, based on sample statistics and probability theory, are used to determine whether a hypothesis is a reasonable statement and should not be rejected or if it is an unreasonable statement and should be rejected. The process of hypothesis testing consists of a series of steps shown in Figure 11.1.

**Figure 11.1: Hypothesis Testing Procedure***




*(*Source:* Wayne W. Daniel and James C. Terrell, Business Statistics, Basic Concepts and Methodology, Houghton Mifflin, Boston, 1997.)

## The Null Hypothesis and Alternative Hypothesis

The **null hypothesis**, designated $H_0$, is the hypothesis that the researcher wants to reject. It is the hypothesis that is actually tested and is the basis for the selection of the test statistics. The null is generally stated as a simple statement about a population parameter. Typical statements of the null hypothesis for the population mean include $H_0$: $\mu = \mu_0$, $H_0$: $\mu \leq \mu_0$, and $H_0$: $\mu \geq \mu_0$, where $\mu$ is the population mean and $\mu_0$ is the hypothesized value of the population mean.

**PROFESSOR'S NOTE**

The null hypothesis always includes the "equal to" condition.

The **alternative hypothesis**, designated $H_a$, is what is concluded if there is sufficient evidence to reject the null hypothesis. It is usually the alternative hypothesis that you are really trying to assess. Why? Since you can never really prove anything with statistics, when the null hypothesis is discredited, the implication is that the alternative hypothesis is valid.

### LOS 11.b: Distinguish between one-tailed and two-tailed tests of hypotheses.

*CFA® Program Curriculum, Volume 1, page 612*

The alternative hypothesis can be one-sided or two-sided. A one-sided test is referred to as a **one-tailed test**, and a two-sided test is referred to as a **two-tailed test**. Whether the test is one- or two-sided depends on the proposition being tested. If a researcher wants to test whether the return on stock options is greater than zero, a one-tailed test should be used. However, a two-tailed test should be used if the research question is whether the return on options is simply different from zero. Two-sided tests allow for deviation on both sides of the hypothesized value (zero). In practice, most hypothesis tests are constructed as two-tailed tests.

A **two-tailed test** for the population mean may be structured as:

$H_0$: $\mu = \mu_0$ versus $H_a$: $\mu \neq \mu_0$

Since the alternative hypothesis allows for values above and below the hypothesized parameter, a two-tailed test uses two **critical values** (or **rejection points**).

The *general decision rule for a two-tailed test* is:

Reject $H_0$ if: test statistic > upper critical value or test statistic < lower critical value

Let's look at the development of the decision rule for a two-tailed test using a *z*-distributed test statistic (a *z*-test) at a 5% level of significance, $\alpha = 0.05$.

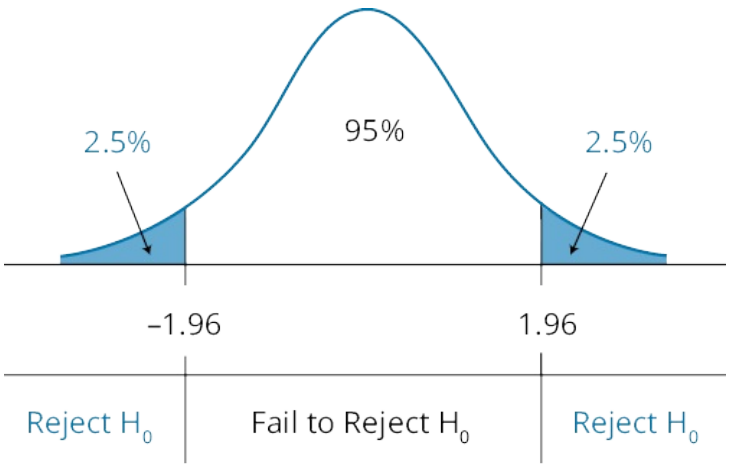
- At $\alpha = 0.05$, the computed test statistic is compared with the critical *z*-values of ±1.96. The values of ±1.96 correspond to $\pm z\alpha_{/2} = \pm z_{0.025}$, which is the range of *z*-values within which 95% of the probability lies. These values are obtained from the cumulative probability table for the standard normal distribution (*z*-table), which is included at the back of this book.
- If the computed test statistic falls outside the range of critical *z*-values (i.e., test statistic > 1.96, or test statistic < –1.96), we reject the null and conclude that the sample statistic is sufficiently different from the hypothesized value.
- If the computed test statistic falls within the range ±1.96, we conclude that the sample statistic is not sufficiently different from the hypothesized value ($\mu = \mu_0$ in this case), and we fail to reject the null hypothesis.

The *decision rule* (rejection rule) *for a two-tailed z-test* at $\alpha = 0.05$ can be stated as:

Reject $H_0$ if: test statistic < –1.96 or test statistic > 1.96

Figure 11.2 shows the standard normal distribution for a two-tailed hypothesis test using the *z*-distribution. Notice that the significance level of 0.05 means that there is 0.05 / 2 = 0.025 probability (area) under each tail of the distribution beyond ±1.96.

**Figure 11.2: Two-Tailed Hypothesis Test Using the Standard Normal (*z*) Distribution**




For a **one-tailed hypothesis test** of the population mean, the null and alternative hypotheses are either:

Upper tail: $H_0$: $\mu \leq \mu_0$ versus $H_a$: $\mu > \mu_0$, or

Lower tail: $H_0$: $\mu \geq \mu_0$ versus $H_a$: $\mu < \mu_0$

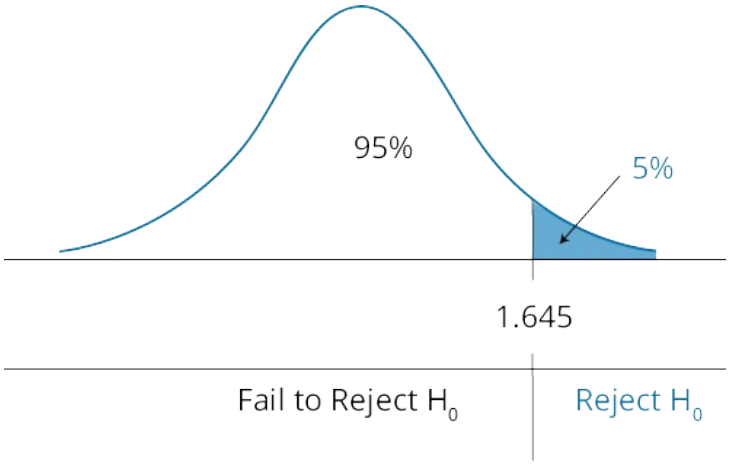
The appropriate set of hypotheses depends on whether we believe the population mean, $\mu$, to be greater than (upper tail) or less than (lower tail) the hypothesized value, $\mu_0$. Using a *z*-test at the 5% level of significance, the computed test statistic is compared with the critical values of 1.645 for the upper tail tests (i.e., $H_a$: $\mu > \mu_0$) or –1.645 for lower tail tests (i.e., $H_a$: $\mu < \mu_0$). These critical values are obtained from a *z*-table, where $-z_{0.05} = -1.645$ corresponds to a cumulative probability equal to 5%, and the $z_{0.05} = 1.645$ corresponds to a cumulative probability of 95% (1 − 0.05).

Let's use the upper tail test structure where $H_0$: $\mu \leq \mu_0$ and $H_a$: $\mu > \mu_0$.

- If the calculated test statistic is greater than 1.645, we conclude that the sample statistic is sufficiently greater than the hypothesized value. In other words, we reject the null hypothesis.
- If the calculated test statistic is less than 1.645, we conclude that the sample statistic is not sufficiently different from the hypothesized value, and we fail to reject the null hypothesis.

Figure 11.3 shows the standard normal distribution and the rejection region for a one-tailed test (upper tail) at the 5% level of significance.

**Figure 11.3: One-Tailed Hypothesis Test Using the Standard Normal (*z*) Distribution**




## The Choice of the Null and Alternative Hypotheses

The most common null hypothesis will be an "equal to" hypothesis. Combined with a "not equal to" alternative, this will require a two-tailed test. The alternative is often the hoped-for hypothesis. The null will include the "equal to" sign and the alternative will include the "not equal to" sign. When the null is that a coefficient is equal to zero, we hope to reject it and show the significance of the relationship.

When the null is less than or equal to, the (mutually exclusive) alternative is framed as greater than, and a one-tail test is appropriate. If we are trying to demonstrate that a return is greater than the risk-free rate, this would be the correct formulation. We will have set up the null and alternative hypothesis so that rejection of the null will lead to acceptance of the alternative, our goal in performing the test. As with a two-tailed test, the null for a one-tailed test will include the "equal to" sign (i.e., either "greater than or equal to" or "less than or equal to"). The alternative will include the opposite sign to the null—either "less than" or "greater than."

**LOS 11.c: Explain a test statistic, Type I and Type II errors, a significance level, and how significance levels are used in hypothesis testing.**

*CFA® Program Curriculum, Volume 1, page 613*

Hypothesis testing involves two statistics: the test statistic calculated from the sample data and the *critical value* of the test statistic. The value of the computed test statistic relative to the critical value is a key step in assessing the validity of a hypothesis.

A test statistic is calculated by comparing the point estimate of the population parameter with the hypothesized value of the parameter (i.e., the value specified in the null hypothesis). With reference to our option return example, this means we are concerned with the difference between the mean return of the sample (i.e., x = 0.001) and the hypothesized mean return (i.e., $\mu_0 = 0$). As indicated in the following expression, the **test statistic** is the difference between the sample statistic and the hypothesized value, scaled by the standard error of the sample statistic.

$$\text{test statistic} = \frac{\text{sample statistic} - \text{hypothesized value}}{\text{standard error of the sample statistic}}$$

The standard error of the sample statistic is the adjusted standard deviation of the sample. When the sample statistic is the sample mean, $\bar{x}$, the standard error of the sample statistic for sample size *n*, is calculated as:

$$\sigma_{\bar{x}} = \sigma / \sqrt{n}$$

when the population standard deviation, σ, *is known*, or

$$s_{\bar{x}} = s / \sqrt{n}$$

when the population standard deviation, σ, *is not known*. In this case, it is estimated using the standard deviation of the sample, *s*.

**PROFESSOR'S NOTE**

Don't be confused by the notation here. A lot of the literature you will encounter in your studies simply uses the term $\sigma_{\bar{x}}$ for the standard error of the test statistic, regardless of whether the population standard deviation or sample standard deviation was used in its computation.

As you will soon see, a test statistic is a random variable that may follow one of several distributions, depending on the characteristics of the sample and the population. We will look at four distributions for test statistics: the *t*-distribution, the *z*-distribution (standard normal distribution), the chi-square distribution, and the *F*-distribution. The critical value for the appropriate test statistic—the value against which the computed test statistic is compared—depends on its distribution.

## Type I and Type II Errors

Keep in mind that hypothesis testing is used to make inferences about the parameters of a given population on the basis of statistics computed for a sample that is drawn from that population. We must be aware that there is some probability that the sample, in some way, does not represent the population, and any conclusion based on the sample about the population may be made in error.

When drawing inferences from a hypothesis test, there are two types of errors:

- **Type I error**: the rejection of the null hypothesis when it is actually true.
- **Type II error**: the failure to reject the null hypothesis when it is actually false.

The **significance level** is the probability of making a Type I error (rejecting the null when it is true) and is designated by the Greek letter alpha (α). For instance, a significance level of 5% (α = 0.05) means there is a 5% chance of rejecting a true null hypothesis. When conducting hypothesis tests, a significance level must be specified in order to identify the critical values needed to evaluate the test statistic.

**LOS 11.d: Explain a decision rule, the power of a test, and the relation between confidence intervals and hypothesis tests.**

*CFA® Program Curriculum, Volume 1, page 615*

The decision for a hypothesis test is to either reject the null hypothesis or fail to reject the null hypothesis. Note that it is statistically incorrect to say "accept" the null hypothesis; it can only be supported or rejected. The **decision rule** for rejecting or failing to reject the null hypothesis is based on the distribution of the test statistic. For example, if the test statistic follows a normal distribution, the decision rule is based on critical values determined from the standard normal distribution (*z*-distribution). Regardless of the appropriate distribution, it must be determined if a one-tailed or two-tailed hypothesis test is appropriate before a decision rule (rejection rule) can be determined.

A decision rule is specific and quantitative. Once we have determined whether a one- or two-tailed test is appropriate, the significance level we require, and the distribution of the test statistic, we can calculate the exact critical value for the test statistic. Then we have a decision rule of the following form: if the test statistic is (greater, less than) the value X, reject the null.

## The Power of a Test

While the significance level of a test is the probability of rejecting the null hypothesis when it is true, the **power of a test** is the probability of correctly rejecting the null hypothesis when it is false. The power of a test is actually one minus the probability of making a Type II error, or 1 – P(Type II error). In other words, the probability of rejecting the null when it is false (power of the test) equals one minus the probability of *not* rejecting the null when it is false (Type II error). When more than one test statistic may be used, the power of the test for the competing test statistics may be useful in deciding which test statistic to use. Ordinarily, we wish to use the test statistic that provides the most powerful test among all possible tests.

Figure 11.4 shows the relationship between the level of significance, the power of a test, and the two types of errors.

**Figure 11.4: Type I and Type II Errors in Hypothesis Testing**

| **Decision** | **True Condition** | |
|---|---|---|
| | **$H_0$ is true** | **$H_0$ is false** |
| Do not reject $H_0$ | Correct decision | Incorrect decision<br>**Type II error** |
| Reject $H_0$ | Incorrect decision<br>**Type I error**<br>Significance level, α, | Correct decision<br>Power of the test<br>= 1 − P(Type II error) |

| | | |
|---|---|---|
| | = P(Type I error) | |

Sample size and the choice of significance level (Type I error probability) will together determine the probability of a Type II error. The relation is not simple, however, and calculating the probability of a Type II error in practice is quite difficult. Decreasing the significance level (probability of a Type I error) from 5% to 1%, for example, will increase the probability of failing to reject a false null (Type II error) and therefore reduce the power of the test. Conversely, for a given sample size, we can increase the power of a test only with the cost that the probability of rejecting a true null (Type I error) increases. For a given significance level, we can decrease the probability of a Type II error and increase the power of a test, only by increasing the sample size.

## The Relation Between Confidence Intervals and Hypothesis Tests

A confidence interval is a range of values within which the researcher believes the true population parameter may lie.

A confidence interval is determined as:

$$\left\{\left[\begin{array}{c}\text{sample}\\ \text{statistic}\end{array}-\left(\begin{array}{c}\text{critical}\\ \text{value}\end{array}\right)\left(\begin{array}{c}\text{standard}\\ \text{error}\end{array}\right)\right]\leq\begin{array}{c}\text{population}\\ \text{parameter}\end{array}\leq\left[\begin{array}{c}\text{sample}\\ \text{statistic}\end{array}+\left(\begin{array}{c}\text{critical}\\ \text{value}\end{array}\right)\right.$$

The interpretation of a confidence interval is that for a level of confidence of 95%, for example, there is a 95% probability that the true population parameter is contained in the interval.

From the previous expression, we see that a confidence interval and a hypothesis test are linked by the critical value. For example, a 95% confidence interval uses a critical value associated with a given distribution at the 5% level of significance. Similarly, a hypothesis test would compare a test statistic to a critical value at the 5% level of significance. To see this relationship more clearly, the expression for the confidence interval can be manipulated and restated as:

–critical value ≤ test statistic ≤ +critical value

This is the range within which we fail to reject the null for a two-tailed hypothesis test at a given level of significance.

**EXAMPLE: Confidence intervals and two-tailed hypothesis tests**

A researcher has gathered data on the daily returns on a portfolio of call options over a recent 250-day period. The mean daily return has been 0.1%, and the sample standard deviation of daily portfolio returns is 0.25%. The researcher believes that the mean daily portfolio return is not equal to zero.

1. Construct a 95% confidence interval for the population mean daily return over the 250-day sample period.
2. Construct a hypothesis test of the researcher's belief.

**Answer:**

1. Given a sample size of 250 with a standard deviation of 0.25%, the standard error can be computed as $s_{\bar{x}} = \frac{s}{\sqrt{n}} = 0.25/\sqrt{250} = 0.0158\%$ .

At the 5% level of significance, the critical $z$-values for the confidence interval are $z_{0.025} = 1.96$ and $-z_{0.025} = -1.96$. Thus, given a sample mean equal to 0.1%, the 95% confidence interval for the population mean is:

$$0.1 - 1.96(0.0158) \leq \mu \leq 0.1 + 1.96(0.0158), \text{ or}$$

$$0.069\% \leq \mu \leq 0.131\%$$

2. First we need to specify the null and alternative hypotheses. The null hypothesis is the one the researcher expects to reject.

$$H_0: \mu_0 = 0 \text{ versus } H_a: \mu_0 \neq 0$$

Since the null hypothesis is an equality, this is a two-tailed test. At a 5% level of significance, the critical $z$-values for a two-tailed test are ±1.96, so the decision rule can be stated as:

Reject $H_0$ if test statistic < –1.96 or test statistic > +1.96

Using the standard error of the sample mean we calculated above, our test statistic is:

$$\frac{0.001}{\left(\frac{0.0025}{\sqrt{250}}\right)} = \frac{0.001}{0.000158} = 6.33$$

Since 6.33 > 1.96, we reject the null hypothesis that the mean daily option return is equal to zero.

Notice the similarity of this analysis with our confidence interval. We rejected the hypothesis $\mu = 0$ because the sample mean of 0.1% is more than 1.96 standard errors from zero. Based on the 95% confidence interval, we reject $\mu = 0$ because zero is more than 1.96 standard errors from the sample mean of 0.1%.

### MODULE QUIZ 11.1

*To best evaluate your performance, enter your quiz answers online.*

1. To test whether the mean of a population is greater than 20, the appropriate null hypothesis is that the population mean is:
   A. less than 20.
   B. greater than 20.
   C. less than or equal to 20.
2. Which of the following statements about hypothesis testing is *most accurate*?
   A. A Type II error is rejecting the null when it is actually true.
   B. The significance level equals one minus the probability of a Type I error.
   C. A two-tailed test with a significance level of 5% has $z$-critical values of ±1.96.
3. For a hypothesis test with a probability of a Type II error of 60% and a probability of a Type I error of 5%, which of the following statements is *most accurate*?
   A. The power of the test is 40%, and there is a 5% probability that the test statistic will exceed the critical value(s).
   B. There is a 95% probability that the test statistic will be between the critical values if this is a two-tailed test.
   C. There is a 5% probability that the null hypothesis will be rejected when actually true, and the probability of rejecting the null when it is false is 40%.
4. If the significance level of a test is 0.05 and the probability of a Type II error is 0.15, what is the power of the test?
   A. 0.850.
   B. 0.950.
   C. 0.975.

## MODULE 11.2: TESTS OF MEANS AND *p*-VALUES

Video covering this content is

### LOS 11.e: Distinguish between a statistical result and an economically meaningful result.

available online.

*CFA® Program Curriculum, Volume 1, page 619*

**Statistical significance** does not necessarily imply **economic significance**. For example, we may have tested a null hypothesis that a strategy of going long all the stocks that satisfy some criteria and shorting all the stocks that do not satisfy the criteria resulted in returns that were less than or equal to zero over a 20-year period. Assume we have rejected the null in favor of the alternative hypothesis that the returns to the strategy are greater than zero (positive). This does not necessarily mean that investing in that strategy will result in economically meaningful positive returns. Several factors must be considered.

One important consideration is transactions costs. Once we consider the costs of buying and selling the securities, we may find that the mean positive returns to the strategy are not enough to generate positive returns. Taxes are another factor that may make a seemingly attractive strategy a poor one in practice. A third reason that statistically significant results may not be economically significant is risk. In the above strategy, we have additional risk from short sales (they may have to be closed out earlier than in the test strategy). Since the statistically significant results were for a period of 20 years, it may be the case that there is significant variation from year to year in the returns from the strategy, even though the mean strategy return is greater than zero. This variation in returns from period to period is an additional risk to the strategy that is not accounted for in our test of statistical significance.

Any of these factors could make committing funds to a strategy unattractive, even though the statistical evidence of positive returns is highly significant. By the nature of statistical tests, a very large sample size can result in highly (statistically) significant results that are quite small in absolute terms.

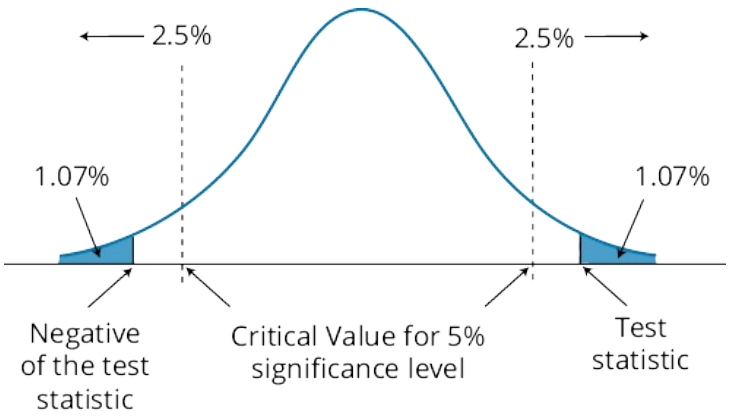
### LOS 11.f: Explain and interpret the *p*-value as it relates to hypothesis testing.

*CFA® Program Curriculum, Volume 1, page 619*

The ***p*-value** is the probability of obtaining a test statistic that would lead to a rejection of the null hypothesis, assuming the null hypothesis is true. It is the smallest level of significance for which the null hypothesis can be rejected. For one-tailed tests, the *p*-value is the probability that lies above the computed test statistic for upper tail tests or below the computed test statistic for lower tail tests. For two-tailed tests, the *p*-value is the probability that lies above the positive value of the computed test statistic *plus* the probability that lies below the negative value of the computed test statistic.

Consider a two-tailed hypothesis test about the mean value of a random variable at the 95% significance level where the test statistic is 2.3, greater than the upper critical value of 1.96. If we consult the *Z*-table, we find the probability of getting a value greater than 2.3 is (1 − 0.9893) = 1.07%. Since it's a two-tailed test, our *p*-value is 2 × 1.07 = 2.14%, as illustrated in Figure 11.5. At a 3%, 4%, or 5% significance level, we would reject the null hypothesis, but at a 2% or 1% significance level we would not. Many researchers report *p*-values without selecting a significance level and allow the reader to judge how strong the evidence for rejection is.

**Figure 11.5: Two-Tailed Hypothesis Test With *p*-Value = 2.14%**




**LOS 11.g: Identify the appropriate test statistic and interpret the results for a hypothesis test concerning the population mean of both large and small samples when the population is normally or approximately normally distributed and the variance is 1) known or 2) unknown.**

*CFA® Program Curriculum, Volume 1, page 620*

When hypothesis testing, the choice between using a critical value based on the *t*-distribution or the *z*-distribution depends on sample size, the distribution of the population, and whether or not the variance of the population is known.

## The *t*-Test

The *t*-test is a widely used hypothesis test that employs a test statistic that is distributed according to a *t*-distribution. Following are the rules for when it is appropriate to use the t-test for hypothesis tests of the population mean.

*Use the* t-*test if the population variance is unknown* and either of the following conditions exist:

- The sample is large ($n \geq 30$).
- The sample is small (less than 30), but the distribution of the population is normal or approximately normal.

If the sample is small and the distribution is nonnormal, we have no reliable statistical test.

The computed value for the test statistic based on the *t*-distribution is referred to as the *t*-statistic. For hypothesis tests of a population mean, a *t*-statistic with n – 1 degrees of freedom is computed as:

$$t_{n-1} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

where:
$\bar{x}$ = sample mean
$\mu_0$ = hypothesized population mean (i.e., the null)
s = standard deviation of the sample
n = sample size

**PROFESSOR'S NOTE**

This computation is not new. It is the same test statistic computation that we have been performing all along. Note the use of the sample standard deviation, s, in the standard error term in the denominator.

To conduct a *t*-test, the *t*-statistic is compared to a critical *t*-value at the desired level of significance with the appropriate degrees of freedom.

In the real world, the underlying variance of the population is rarely known, so the *t*-test enjoys widespread application.

## The *z*-Test

The *z*-test is the appropriate hypothesis test of the population mean when the *population is normally distributed with known variance*. The computed test statistic used with the *z*-test is referred to as the *z*-statistic. The *z*-statistic for a hypothesis test for a population mean is computed as follows:

$$\text{z-statistic} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

where:
$\bar{x}$ = sample mean
$\mu_0$ = hypothesized population mean
$\sigma$ = standard deviation of the *population*
n = sample size

To test a hypothesis, the *z*-statistic is compared to the critical *z*-value corresponding to the significance of the test. Critical *z*-values for the most common levels of significance are displayed in Figure 11.6. You should have these memorized by now.

**Figure 11.6: Critical z-Values**

| Level of Significance | Two-Tailed Test | One-Tailed Test |
|---|---|---|
| 0.10 = 10% | ±1.65 | +1.28 or –1.28 |
| 0.05 = 5% | ±1.96 | +1.65 or –1.65 |
| 0.01 = 1% | ±2.58 | +2.33 or –2.33 |

When the *sample size is large* and the *population variance is unknown*, the *z*-statistic is:

$$\text{z-statistic} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

where:
$\bar{x}$ = sample mean
$\mu_0$ = hypothesized population mean
s = standard deviation of the *sample*
n = sample size

Note the use of the sample standard deviation, *s*, versus the population standard deviation, $\sigma$. Remember, this is acceptable if the sample size is large, although the *t*-statistic is the more conservative measure when the population variance is unknown.

**EXAMPLE: *z*-test or *t*-test?**

Referring to our previous option portfolio mean return example, determine which test statistic (*z* or *t*) should be used.

**Answer:**

The population variance for our sample of returns is unknown. Hence, the *t*-distribution is appropriate. With 250 observations, however, the sample is considered to be large, so the *z*-distribution would also be acceptable. Because our sample is so large, the critical values for the *t* and *z* are almost identical. Hence, there is almost no difference in the likelihood of rejecting a true null.

**EXAMPLE: The *z*-test**

When your company's gizmo machine is working properly, the mean length of gizmos is 2.5 inches. However, from time to time the machine gets out of alignment and produces gizmos that are either too long or too short. When this happens, production is stopped and the machine is adjusted. To check the machine, the quality control department takes a gizmo sample each day. Today, a random sample of 49 gizmos showed a mean length of 2.49 inches. The population standard deviation is known to be 0.021 inches. Using a 5% significance level, determine if the machine should be shut down and adjusted.

**Answer:**

Let μ be the mean length of all gizmos made by this machine, and let x be the corresponding mean for the sample.

Let's follow the hypothesis testing procedure presented earlier in Figure 11.1. Again, you should know this process!

*Statement of hypothesis*. For the information provided, the null and alternative hypotheses are appropriately structured as:

$H_0$: μ = 2.5 (The machine does not need an adjustment.)

$H_a$: μ ≠ 2.5 (The machine needs an adjustment.)

Note that since this is a two-tailed test, $H_a$ allows for values above and below 2.5.

*Select the appropriate test statistic*. Since the population variance is known and the sample size is > 30, the *z*-statistic is the appropriate test statistic. The *z*-statistic is computed as:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

*Specify the level of significance*. The level of significance is given at 5%, implying that we are willing to accept a 5% probability of rejecting a true null hypothesis.

*State the decision rule regarding the hypothesis*. The ≠ sign in the alternative hypothesis indicates that the test is two-tailed with two rejection regions, one in each tail of the standard normal distribution curve. Because the total area of both rejection regions combined is 0.05 (the significance level), the area of the rejection region in each tail is 0.025. You should know that the critical *z*-values for $\pm z_{0.025}$ are ±1.96. This means that the null hypothesis should not be rejected if the computed *z*-statistic lies between –1.96 and +1.96 and should be rejected if it lies outside of these critical values. The decision rule can be stated as:

Reject $H_0$ if $-z_{0.025} > z\text{-statistic} > z_{0.025}$, or equivalently,

Reject $H_0$ if: –1.96 > *z*-statistic > + 1.96

*Collect the sample and calculate the test statistic*. The value of x from the sample is 2.49. Since σ is given as 0.021, we calculate the *z*-statistic using σ as follows:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{2.49 - 2.5}{0.021/\sqrt{49}} = \frac{-0.01}{0.003} = -3.33$$

*Make a decision regarding the hypothesis*. The calculated value of the *z*-statistic is –3.33. Since this value is less than the critical value, $-z_{0.025} = -1.96$, it falls in the rejection region in the left tail of the z-distribution. Hence, there is sufficient evidence to reject $H_0$.

*Make a decision based on the results of the test*. Based on the sample information and the results of the test, it is concluded that the machine is out of adjustment and should be shut down for repair.

**EXAMPLE: One-tailed test**

Using the data from the previous example and a 5% significance level, test the hypothesis that the mean length of gizmos is less than 2.5 inches.

**Answer:**

In this case, we use a one-tailed test with the following structure:

$H_0$: $\mu \geq 2.5$ versus $H_a$: $\mu < 2.5$

The appropriate decision rule for this one-tailed test at a significance level of 5% is:

Reject $H_0$ if test statistic < –1.645

The test statistic is computed in the same way, regardless of whether we are using a one-tailed or a two-tailed test. From the previous example, we know that the test statistic for the gizmo sample is –3.33. Because –3.33 < –1.645, we can reject the null hypothesis and conclude that the mean length is statistically less than 2.5 at a 5% level of significance.

## MODULE QUIZ 11.2

*To best evaluate your performance, enter your quiz answers online.*

1. Using historical data, a hedge fund manager designs a test of whether abnormal returns are positive on average. The test results in a *p*-value of 3%. The manager can *most appropriately*:
   A. reject the hypothesis that abnormal returns are less than or equal to zero, using a 1% significance level.
   B. reject the hypothesis that abnormal returns are less than or equal to zero, using a 5% significance level.
   C. conclude that the strategy produces positive abnormal returns on average, using a 5% significance level.
2. An analyst wants to test a hypothesis concerning the population mean of monthly returns for a composite that has existed for 24 months. The analyst may appropriately use:
   A. a *t*-test but not a *z*-test if returns for the composite are normally distributed.
   B. either a *t*-test or a *z*-test if returns for the composite are normally distributed.
   C. a *t*-test but not a *z*-test, regardless of the distribution of returns for the composite.

**Use the following segment of Student's *t*-distribution for Question 3.**

| Level of Significance for One-Tailed Test | | | | |
|---|---|---|---|---|
| df | 0.100 | 0.050 | 0.025 | 0.01 |

| Level of Significance for Two-Tailed Test | | | | |
|---|---|---|---|---|
| df | 0.20 | 0.10 | 0.05 | 0.02 |
| 11 | 1.363 | 1.796 | 2.201 | 2.718 |
| 12 | 1.356 | 1.782 | 2.179 | 2.681 |
| 13 | 1.350 | 1.771 | 2.160 | 2.650 |
| 14 | 1.345 | 1.761 | 2.145 | 2.624 |
| 15 | 1.341 | 1.753 | 2.131 | 2.602 |

3. From a sample of 14 observations, an analyst calculates a t-statistic to test a hypothesis that the population mean is equal to zero. If the analyst chooses a 5% significance level, the

appropriate critical value is:

A. less than 1.80.
B. greater than 2.15.
C. between 1.80 and 2.15.

## MODULE 11.3: MEAN DIFFERENCES, DIFFERENCE IN MEANS

Video covering this content is available online.

**LOS 11.h: Identify the appropriate test statistic and interpret the results for a hypothesis test concerning the equality of the population means of two at least approximately normally distributed populations, based on independent random samples with 1) equal or 2) unequal assumed variances.**

*CFA® Program Curriculum, Volume 1, page 628*

Up to this point, we have been concerned with tests of a single population mean. In practice, we frequently want to know if there is a difference between the means of two populations. There are two *t*-tests that are used to test differences between the means of two populations. Application of either of these tests requires that we are reasonably certain that our samples are independent and that they are taken from two populations that are normally distributed. Both of these *t*-tests can be used when the population variance is unknown. In one case, the population variances are assumed to be equal, and the sample observations are pooled. In the other case, however, no assumption is made regarding the equality between the two population variances, and the *t*-test uses an approximated value for the degrees of freedom.

### PROFESSOR'S NOTE

Please note the language of the LOS here. Candidates must "Identify the appropriate test statistic and interpret the results of a hypothesis test…." Certainly you should know that this is a *t*-test, and that we reject the hypothesis of equality when the test statistic is outside the critical *t*-values. Don't worry about memorizing the following formulas. You should understand, however, that we can pool the data to get the standard deviation of the difference in means when we assume equal variances, while both sample variances are used to get the standard error of the difference in means when we assume the population variances are not equal.

A pooled variance is used with the *t*-test for testing the hypothesis that the means of two normally distributed populations are equal, when the variances of the populations are **unknown but assumed to be equal**.

Assuming independent samples, the *t*-statistic in this case is computed as:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\left(\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}\right)^{1/2}}$$

where:

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$s_1^2$ = variance of the first sample

$s_2^2$ = variance of the second sample

$n_1$ = number of observations in the first sample

$n_2$ = number of observations in the second sample

*Note:* The degrees of freedom, df, is ($n_1 + n_2 - 2$).

Since we assume that the variances are equal, we just add the variances of the two sample means in order to calculate the standard error in the denominator.

The *t*-test for equality of population means when the populations are normally distributed and have **variances that are unknown and assumed to be unequal** uses the sample variances for both populations. Assuming independent samples, the *t*-statistic in this case is computed as follows:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^{1/2}}$$

where:

$$\text{degrees of freedom} = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1} + \frac{\left(s_2^2/n_2\right)^2}{n_2}}$$

and where:

$s_1^2$ = variance of the first sample

$s_2^2$ = variance of the second sample

$n_1$ = number of observations in the first sample

$n_2$ = number of observations in the second sample

With no assumption of equal variances, the denominator (standard error) is based on the individual sample variances for each sample.

You do not need to memorize these formulas, but should understand the numerator, the fact that these are *t*-statistics, and that the variance of the pooled sample is used when the sample variances are assumed to be equal. The intuition here is straightforward. If the sample means are very close together, the numerator of the *t*-statistic (and the *t*-statistic itself) are small, and we do not reject equality. If the sample means are far apart, the numerator of the *t*-statistic (and the *t*-statistic itself) are large, and we reject equality. Perhaps not as easy to remember is the fact that this test is only valid for two populations that are independent and normally distributed.

**EXAMPLE: Difference between means – equal variances**

Sue Smith is investigating whether the abnormal returns for acquiring firms during merger announcement periods differ for horizontal and vertical mergers. She estimates the abnormal returns for a sample of acquiring firms associated with horizontal mergers and a sample of acquiring firms involved in vertical mergers. Smith finds that abnormal returns from horizontal mergers have a mean of 1.0% and a standard deviation of 1.0%, while abnormal returns from vertical mergers have a mean of 2.5% and a standard deviation of 2.0%.

Smith assumes that the samples are independent, the population means are normally distributed, and the population variances are equal.

Smith calculates the *t*-statistic as –5.474 and the degrees of freedom as 120. Using a 5% significance level, should Smith reject or fail to reject the null hypothesis that the abnormal returns to acquiring firms during the announcement period are the same for horizontal and vertical mergers?

**Answer:**

Since this is a two-tailed test, the structure of the hypotheses takes the following form:

$H_0$: $\mu_1 - \mu_2 = 0$ versus $H_a$: $\mu_1 - \mu_2 \neq 0$

where:

$\mu_1$ = the mean of the abnormal returns for the horizontal mergers

$\mu_2$ = the mean of the abnormal returns for the vertical mergers

From the following *t*-table segment, the critical *t*-value for a 5% level of significance at $\alpha / 2 = p = 0.025$ with df = 120, is 1.980.

**Partial *t*-Table**

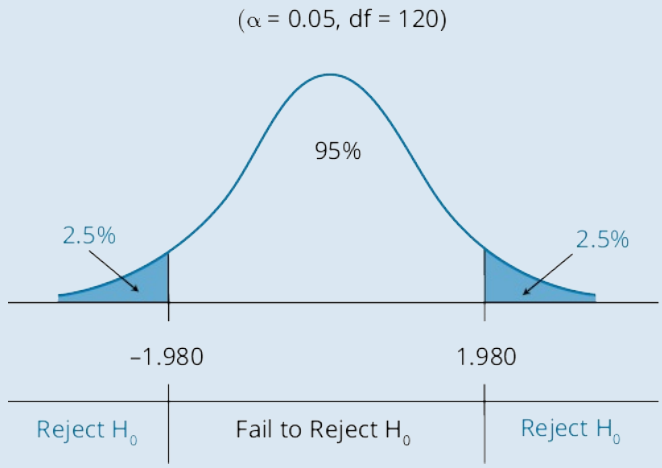
| df | One-Tailed Probabilities (p) | | |
|---|---|---|---|
| | **p = 0.10** | **p = 0.05** | **p = 0.025** |
| 110 | 1.289 | 1.659 | 1.982 |
| 120 | 1.289 | 1.658 | 1.980 |
| 200 | 1.286 | 1.653 | 1.972 |

Thus, the decision rule can be stated as:

Reject $H_0$ if *t*-statistic < –1.980 or *t*-statistic > 1.980

The rejection region for this test is illustrated in the following figure.

**Decision Rule for Two-Tailed *t*-Test**




Since the test statistic, –5.474, falls to the left of the lower critical *t*-value, Smith can reject the null hypothesis and conclude that mean abnormal returns are different for horizontal and vertical mergers.

## LOS 11.i: Identify the appropriate test statistic and interpret the results for a hypothesis test concerning the mean difference of two normally distributed populations.

*CFA® Program Curriculum, Volume 1, page 632*

While the tests considered in the previous section were of the difference between the means of two independent samples, sometimes our samples may be dependent. If the observations in the two samples both depend on some other factor, we can construct a "paired comparisons" test of whether the means of the differences between observations for the two samples are different. Dependence may result from an event that affects both sets of observations for a

number of companies or because observations for two firms over time are both influenced by market returns or economic conditions.

For an example of a paired comparisons test, consider a test of whether the returns on two steel firms were equal over a 5-year period. We can't use the difference in means test because we have reason to believe that the samples are not independent. To some extent, both will depend on the returns on the overall market (market risk) and the conditions in the steel industry (industry specific risk). In this case, our pairs will be the returns on each firm over the same time periods, so we use the differences in monthly returns for the two companies. The paired comparisons test is just a test of whether the average difference between monthly returns is significantly different from zero, based on the standard error of the differences in monthly returns.

Remember, the paired comparisons test also requires that the sample data be normally distributed. Although we frequently just want to test the hypothesis that the mean of the differences in the pairs is zero ($\mu_{dz} = 0$), the general form of the test for any hypothesized mean difference, $\mu_{dz}$, is as follows:

$H_0$: $\mu_d = \mu_{dz}$ versus $H_a$: $\mu_d \neq \mu_{dz}$

where:
$\mu_d$ = mean of the population of paired differences
$\mu_{dz}$ = hypothesized mean of paired differences, which is commonly zero

For one-tail tests, the hypotheses are structured as either:

$H_0$: $\mu_d \leq \mu_{dz}$ versus $H_a$: $\mu_d > \mu_{dz}$, or $H_0$: $\mu_d \geq \mu_{dz}$ versus $H_a$: $\mu_d < \mu_{dz}$

For the paired comparisons test, the *t*-statistic with n − 1 degrees of freedom is computed as:

$$t = \frac{\bar{d} - \mu_{dz}}{s_{\bar{d}}}$$

where:

$$\bar{d} = \text{sample mean difference} = \frac{1}{n}\sum_{i=1}^{n} d_i$$

$d_i$ = difference between the *i*th pair of observations

$$s_{\bar{d}} = \text{standard error of the mean difference} = \frac{s_d}{\sqrt{n}}$$

$$s_d = \text{sample standard deviation} = \left(\frac{\sum_{i=1}^{n}(d_i - \bar{d})^2}{n-1}\right)^{1/2}$$

n = the number of paired observations

**EXAMPLE: Paired comparisons test**

Joe Andrews is examining changes in estimated betas for the common stock of companies in the telecommunications industry before and after deregulation. Andrews believes that the betas may decline because of deregulation since companies are no longer subject to the uncertainties of rate regulation or that they may increase because there is more uncertainty regarding competition in the industry. Andrews calculates a *t*-statistic of 10.26 for this hypothesis test, based on a sample size of 39. Using a 5% significance level, determine whether there is a change in betas.

**Answer:**

Because the mean difference may be positive or negative, a two-tailed test is in order here. Thus, the hypotheses are structured as:

$H_0$: $\mu_d = 0$ versus $H_a$: $\mu_d \neq 0$

There are 39 – 1 = 38 degrees of freedom. Using the t-distribution, the two-tailed critical *t*-values for a 5% level of significance with df = 38 is ±2.024. As indicated in the following table, the critical *t*-value of 2.024 is located at the intersection of the p = 0.025 column and the df = 38 row. The one-tailed probability of 0.025 is used because we need 2.5% in each tail for 5% significance with a two-tailed test.

**Partial *t*-Table**

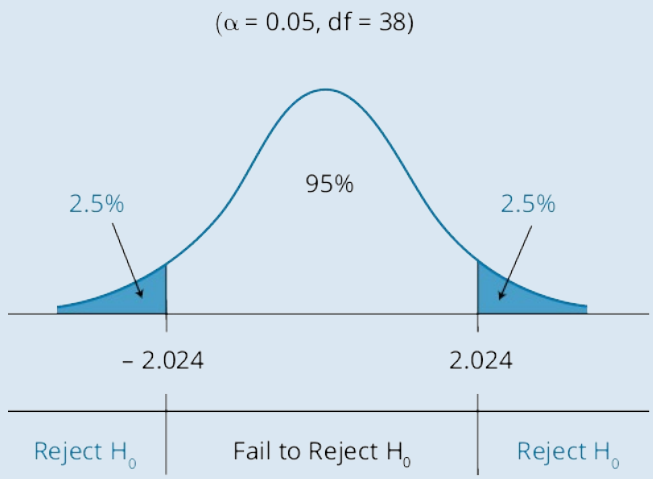
| df | One-Tailed Probabilities (p) | | |
|---|---|---|---|
| | **p = 0.10** | **p = 0.05** | **p = 0.025** |
| 38 | 1.304 | 1.686 | **2.024** |
| 39 | 1.304 | 1.685 | 2.023 |
| 40 | 1.303 | 1.684 | 2.021 |

Thus, the decision rule becomes:

Reject $H_0$ if *t*-statistic < –2.024, or *t*-statistic > 2.024

This decision rule is illustrated in the following figure.

**Decision Rule for a Two-Tailed Paired Comparisons Test**




The test statistic, 10.26, is greater than the critical *t*-value, 2.024—it falls in the rejection region to the right of 2.024 in the previous figure. Thus, we reject the null hypothesis of no difference, concluding that there *is* a statistically significant difference between mean firm betas before and after deregulation.

Keep in mind that we have been describing two distinct hypothesis tests, one about the significance of the difference between the means of two populations and one about the significance of the mean of the differences between pairs of observations. Here are rules for when these tests may be applied:

- The test of the differences in means is used when there are two *independent samples*.
- A test of the significance of the mean of the differences between paired observations is used when the samples are *not independent*.

**PROFESSOR'S NOTE**

The LOS here say "Identify the appropriate test statistic and interpret the results..." I can't believe candidates are expected to memorize these formulas (or that you would be a better analyst if you did). You should instead focus on the fact that both of these tests involve *t*-statistics and depend on the degrees of freedom. Also note that when samples are independent, you can use the difference in means test, and when they are dependent, we must use the paired comparison (mean differences) test. In that case, with a null hypothesis that there is no difference in means, the test statistic is simply the mean of the differences between each pair of observations, divided by the standard error of those differences.

## LOS 11.j: Identify the appropriate test statistic and interpret the results for a hypothesis test concerning 1) the variance of a normally distributed population, and 2) the equality of the variances of two normally distributed populations based on two independent random samples.

*CFA® Program Curriculum, Volume 1, page 636*

The *chi-square test* is used for hypothesis tests concerning the variance of a normally distributed population. Letting $\sigma^2$ represent the true population variance and $\sigma_0^2$ represent the hypothesized variance, the hypotheses for a two-tailed test of a single population variance are structured as:

$H_0$: $\sigma^2 = \sigma_0^2$ versus $H_a$: $\sigma^2 \neq \sigma_0^2$

The hypotheses for one-tailed tests are structured as:

$H_0$: $\sigma^2 \leq \sigma_0^2$ versus $H_a$: $\sigma^2 >, \sigma_0^2$ or

$H_0$: $\sigma^2 \geq \sigma_0^2$ versus $H_a$: $\sigma^2 < \sigma_0^2$

Hypothesis testing of the population variance requires the use of a chi-square distributed test statistic, denoted $\chi^2$. The chi-square distribution is asymmetrical and approaches the normal distribution in shape as the degrees of freedom increase.

To illustrate the chi-square distribution, consider a two-tailed test with a 5% level of significance and 30 degrees of freedom. As displayed in Figure 11.7, the critical chi-square values are 16.791 and 46.979 for the lower and upper bounds, respectively. These values are obtained from a chi-square table, which is used in the same manner as a *t*-table. A portion of a chi-square table is presented in Figure 11.8.

Note that the chi-square values in Figure 11.8 correspond to the probabilities in the right tail of the distribution. As such, the 16.791 in Figure 11.7 is from the column headed *0.975* because 95% + 2.5% of the probability is to the right of it. The 46.979 is from the column headed *0.025* because only 2.5% probability is to the right of it. Similarly, at a 5% level of significance with 10 degrees of freedom, Figure 11.8 shows that the critical chi-square values for a two-tailed test are 3.247 and 20.483.

**Figure 11.7: Decision Rule for a Two-Tailed Chi-Square Test**

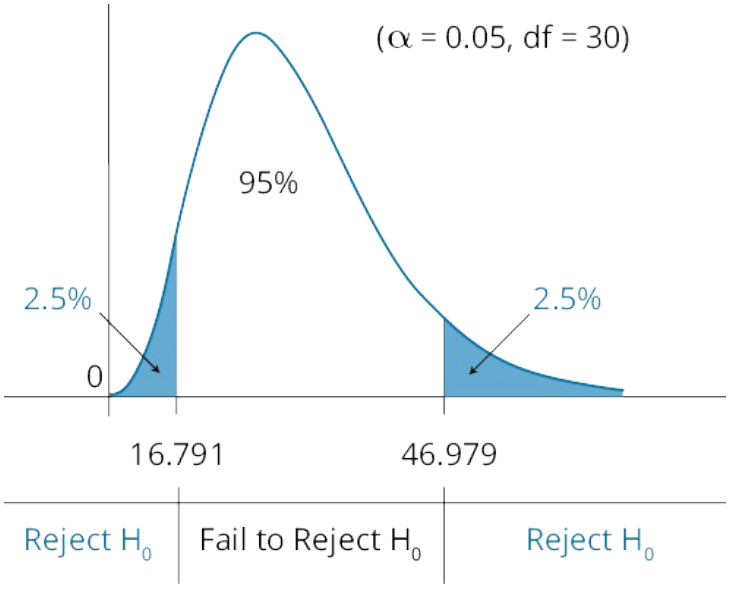



**Figure 11.8: Chi-Square Table**

| Degrees of Freedom | Probability in Right Tail | | | | | |
|---|---|---|---|---|---|---|
| | **0.975** | **0.95** | **0.90** | **0.1** | **0.05** | **0.025** |
| 9 | 2.700 | 3.325 | 4.168 | 14.684 | 16.919 | 19.023 |
| 10 | 3.247 | 3.940 | 4.865 | 15.987 | 18.307 | 20.483 |
| 11 | 3.816 | 4.575 | 5.578 | 17.275 | 19.675 | 21.920 |
| 30 | 16.791 | 18.493 | 20.599 | 40.256 | 43.773 | 46.979 |

The chi-square test statistic, $\chi^2$, with n − 1 degrees of freedom, is computed as:

$$\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma_0^2}$$

where:
n = sample size
$s^2$ = sample variance
$\sigma_0^2$ = hypothesized value for the population variance.

Similar to other hypothesis tests, the chi-square test compares the test statistic, $\chi^2_{n-1}$, to a critical chi-square value at a given level of significance and n − 1 degrees of freedom. Note that since the chi-square distribution is bounded below by zero, chi-square values cannot be negative.

**EXAMPLE: Chi-square test for a single population variance**

Historically, High-Return Equity Fund has advertised that its monthly returns have a standard deviation equal to 4%. This was based on estimates from the 2005–2013 period. High-Return wants to verify whether this claim still adequately describes the standard deviation of the fund's returns. High-Return collected monthly returns for the 24-month period between 2013 and 2015 and measured a standard deviation of monthly returns of 3.8%. High-Return calculates a test statistic of 20.76. Using a 5% significance level, determine if the more recent standard deviation is different from the advertised standard deviation.

**Answer:**

The null hypothesis is that the standard deviation is equal to 4% and, therefore, the variance of monthly returns for the population is $(0.04)^2 = 0.0016$. Since High-Return simply wants to test whether the standard deviation has changed, up or down, a two-sided test should be used. The hypothesis test structure takes the form:

$$H_0: \sigma_0^2 = 0.0016 \text{ versus } H_a: \sigma^2 \neq 0.0016$$

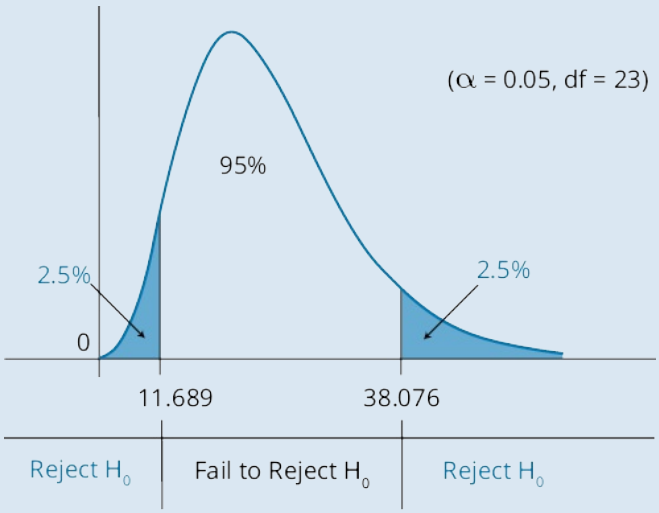
The appropriate test statistic for tests of variance is a chi-square statistic.

With a 24-month sample, there are 23 degrees of freedom. Using the table of chi-square values in Appendix E at the back of this book, for 23 degrees of freedom and probabilities of 0.975 and 0.025, we find two critical values, 11.689 and 38.076. Thus, the decision rule is:

$$\text{Reject } H_0 \text{ if } \chi^2 < 11.689, \text{ or } \chi^2 > 38.076$$

This decision rule is illustrated in the following figure.

**Decision Rule for a Two-Tailed Chi-Square Test of a Single Population Variance**




Since the computed test statistic, $\chi^2$, falls between the two critical values, we cannot reject the null hypothesis that the variance is equal to 0.0016. The recently measured standard deviation is close enough to the advertised standard deviation that we cannot say that it is different from 4%, at a 5% level of significance.

## Testing the Equality of the Variances of Two Normally Distributed Populations, Based on Two Independent Random Samples

The hypotheses concerned with the equality of the variances of two populations are tested with an *F*-distributed test statistic. Hypothesis testing using a test statistic that follows an *F*-distribution is referred to as the *F*-test. The *F*-test is used under the assumption that the populations from which samples are drawn are normally distributed and that the samples are independent.

If we let $\sigma_1^2$ and $\sigma_2^2$ represent the variances of normal Population 1 and Population 2, respectively, the hypotheses for the two-tailed *F*-test of differences in the variances can be structured as:

$$H_0: \sigma_1^2 = \sigma_2^2 \text{ versus } H_a: \sigma_1^2 \neq \sigma_2^2$$

and the one-sided test structures can be specified as:

$H_0$: $\sigma_1^2 \leq \sigma_2^2$ versus $H_a$: $\sigma_1^2 > \sigma_2^2$, or $H_0$: $\sigma_1^2 \geq \sigma_2^2$ versus $H_a$: $\sigma_1^2 < \sigma_2^2$

The test statistic for the *F*-test is the ratio of the sample variances. The *F*-statistic is computed as:

$$F = \frac{s_1^2}{s_2^2}$$

where:
$s_1^2$ = variance of the sample of $n_1$ observations drawn from Population 1
$s_2^2$ = variance of the sample of $n_2$ observations drawn from Population 2

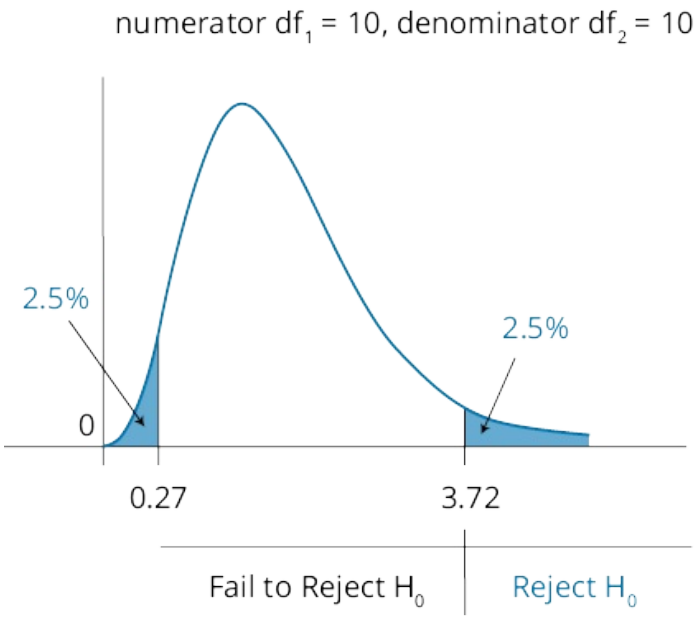
Note that $n_1 - 1$ and $n_2 - 1$ are the degrees of freedom used to identify the appropriate critical value from the *F*-table (provided in the Appendix).

**PROFESSOR'S NOTE**

Always put the larger variance in the numerator ($s_1^2$ ). Following this convention means we only have to consider the critical value for the right-hand tail.

An *F*-distribution is presented in Figure 11.9. As indicated, the *F*-distribution is right-skewed and is bounded by zero on the left-hand side. The shape of the *F*-distribution is determined by *two separate degrees of freedom*, the numerator degrees of freedom, $df_1$, and the denominator degrees of freedom, $df_2$.

**Figure 11.9: F-Distribution**




Note that when the sample variances are equal, the value of the test statistic is 1. The upper critical value is always greater than one (the numerator is significantly greater than the denominator), and the lower critical value is always less than one (the numerator is significantly smaller than the denominator). In fact, the lower critical value is the reciprocal of the upper critical value. For this reason, in practice we put the larger sample variance in the numerator and consider only the upper critical value.

**EXAMPLE: *F*-test for equal variances**

Annie Cower is examining the earnings for two different industries. Cower suspects that the variance of earnings in the textile industry is different from the variance of earnings in the paper industry. To confirm this suspicion, Cower has looked at a sample of 31 textile manufacturers and a sample of 41 paper companies. She measured the sample standard deviation of earnings across the textile industry to be $4.30 and that of the paper industry companies to be $3.80. Cower calculates a test statistic of 1.2805. Using a 5% significance level, determine if the earnings of the textile industry have greater standard deviation than those of the paper industry.

**Answer:**

In this example, we are concerned with whether the variance of earnings for companies in the textile industry is equal to the variance of earnings for companies in the paper industry. As such, the test hypotheses can be appropriately structured as:

$$H_0: \sigma_1^2 = \sigma_2^2 \text{ versus } H_a: \sigma_1^2 \neq \sigma_2^2$$

For tests of difference between variances, the appropriate test statistic is:

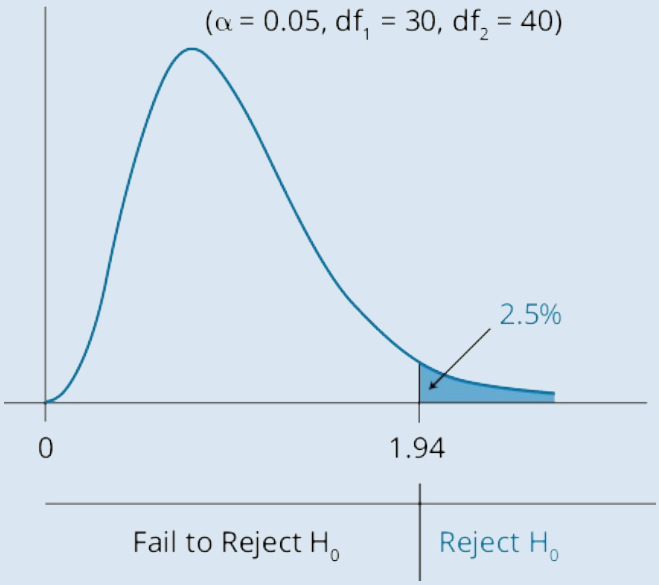
$$F = \frac{s_1^2}{s_2^2}$$

where $s_1^2$ is the larger sample variance.

Using the sample sizes for the two industries, the critical *F*-value for our test is found to be 1.94. This value is obtained from the table of the *F*-distribution for 2.5% in the upper tail, with $df_1 = 30$ and $df_2 = 40$. Thus, if the computed *F*-statistic is greater than the critical value of 1.94, the null hypothesis is rejected. The decision rule, illustrated in the following figure, can be stated as:

Reject $H_0$ if $F > 1.94$

Decision Rule for *F*-Test




Since the calculated *F*-statistic of 1.2805 is less than the critical *F*-statistic of 1.94, Cower cannot reject the null hypothesis. Cower should conclude that the earnings variances of the industries are not significantly different from one another at a 5% level of significance.

## LOS 11.k: Formulate a test of the hypothesis that the population correlation coefficient equals zero and determine whether the hypothesis is rejected at a given level of significance.

*CFA® Program Curriculum, Volume 1, page 642*

Correlation measures the strength of the relationship between two variables. If the correlation between two variables is zero, there is no linear relationship between them. When the sample correlation coefficient for two variables is different from zero, we must address the question of whether the true population correlation coefficient ($\rho$) is equal to zero. The appropriate test statistic for the hypothesis that the population correlation equals zero, when the two variables are normally distributed, is:

$\frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$, where r = sample correlation and n = sample size. This test statistic follows a *t*-distribution with n – 2 degrees of freedom. It is worth noting that the test statistic increases, not only with the sample correlation coefficient, but also with sample size.

**EXAMPLE: Test of the hypothesis that the population correlation coefficient equals zero**

A researcher computes the sample correlation coefficient for two normally distributed random variables as 0.35, based on a sample size of 42. Determine whether to reject the hypothesis that the population correlation coefficient is equal to zero at a 5% significance level.

**Answer:**

Our test statistic is $\frac{0.35\sqrt{42-2}}{\sqrt{1-0.35^2}} = 2.363$. Using the t-table with 42 – 2 = 40 degrees of freedom for a two-tailed test and a significance level of 5%, we can find the critical value of 2.021. Because our computed test statistic is of 2.363 is greater than 2.021, we reject the hypothesis that the population mean is zero and conclude that it is not equal to zero. That is, the two populations are correlated, in this case positively.

**PROFESSOR'S NOTE**

The correlation coefficient we refer to here is the Pearson correlation coefficient, which is a measure of the linear relationship between two variables. There are other correlation coefficients that better measure the strength of any non-linear relationship between two variables.

## LOS 11.l: Distinguish between parametric and nonparametric tests and describe situations in which the use of nonparametric tests may be appropriate.

*CFA® Program Curriculum, Volume 1, page 643*

**Parametric tests** rely on assumptions regarding the distribution of the population and are specific to population parameters. For example, the *z*-test relies upon a mean and a standard deviation to define the normal distribution. The *z*-test also requires that either the sample is large, relying on the central limit theorem to assure a normal sampling distribution, or that the population is normally distributed.

**Nonparametric tests** either do not consider a particular population parameter or have few assumptions about the population that is sampled. Nonparametric tests are used when there is concern about quantities other than the parameters of a distribution or when the assumptions of parametric tests can't be supported. They are also used when the data are not suitable for parametric tests (e.g., ranked observations).

Situations where a nonparametric test is called for are the following:

1. The assumptions about the distribution of the random variable that support a parametric test are not met. An example would be a hypothesis test of the mean value for a

variable that comes from a distribution that is not normal and is of small size so that neither the *t*-test nor the *z*-test is appropriate.

2. When data are ranks (an ordinal measurement scale) rather than values.
3. The hypothesis does not involve the parameters of the distribution, such as testing whether a variable is normally distributed. We can use a nonparametric test, called a runs test, to determine whether data are random. A runs test provides an estimate of the probability that a series of changes (e.g., +, +, –, –, +, –,….) are random.

The **Spearman rank correlation test** can be used when the data are not normally distributed. Consider the performance ranks of 20 mutual funds for two years. The ranks (1 through 20) are not normally distributed, so a standard *t*-test of the correlations is not appropriate. A large positive value of the Spearman rank correlations, such as 0.85, would indicate that a high (low) rank in one year is associated with a high (low) rank in the second year. Alternatively, a large negative rank correlation would indicate that a high rank in year 1 suggests a low rank in year 2, and vice versa.

## MODULE QUIZ 11.3

*To best evaluate your performance, enter your quiz answers online.*

1. Which of the following assumptions is *least likely* required for the difference in means test based on two samples?
   A. The two samples are independent.
   B. The two populations are normally distributed.
   C. The two populations have equal variances.
2. William Adams wants to test whether the mean monthly returns over the last five years are the same for two stocks. If he assumes that the returns distributions are normal and have equal variances, the type of test and test statistic are *best* described as:
   A. paired comparisons test, *t*-statistic.
   B. paired comparisons test, *F*-statistic.
   C. difference in means test, *t*-statistic.
3. Which of the following statements about the *F*-distribution and chi-square distribution is *least accurate*? Both distributions:
   A. are asymmetrical.
   B. are bounded by zero on the left.
   C. have means that are less than their standard deviations.
4. The appropriate test statistic for a test of the equality of variances for two normally distributed random variables, based on two independent random samples, is:
   A. the *t*-test.
   B. the *F*-test.
   C. the $\chi^2$ test.
5. The appropriate test statistic to test the hypothesis that the variance of a normally distributed population is equal to 13 is:
   A. the *t*-test.
   B. the *F*-test.
   C. the $\chi^2$ test.

## KEY CONCEPTS

### LOS 11.a

The hypothesis testing process requires a statement of a null and an alternative hypothesis, the selection of the appropriate test statistic, specification of the significance level, a decision rule, the calculation of a sample statistic, a decision regarding the hypotheses based on the test, and a decision based on the test results.

The null hypothesis is what the researcher wants to reject. The alternative hypothesis is what the researcher wants to support, and it is accepted when the null hypothesis is rejected.

### LOS 11.b

A two-tailed test results from a two-sided alternative hypothesis (e.g., $H_a$: $\mu \neq \mu_0$). A one-tailed test results from a one-sided alternative hypothesis (e.g., $H_a$: $\mu > \mu_0$, or $H_a$: $\mu < \mu_0$).

### LOS 11.c

The test statistic is the value that a decision about a hypothesis will be based on. For a test about the value of the mean of a distribution:

$$\text{test statistic} = \frac{\text{sample mean} - \text{hypothesized mean}}{\text{standard error of sample mean}}$$

A Type I error is the rejection of the null hypothesis when it is actually true, while a Type II error is the failure to reject the null hypothesis when it is actually false.

The significance level can be interpreted as the probability that a test statistic will reject the null hypothesis by chance when it is actually true (i.e., the probability of a Type I error). A significance level must be specified to select the critical values for the test.

### LOS 11.d

Hypothesis testing compares a computed test statistic to a critical value at a stated level of significance, which is the decision rule for the test.

The power of a test is the probability of rejecting the null when it is false. The power of a test = 1 − P(Type II error).

A hypothesis about a population parameter is rejected when the sample statistic lies outside a confidence interval around the hypothesized value for the chosen level of significance.

### LOS 11.e

Statistical significance does not necessarily imply economic significance. Even though a test statistic is significant statistically, the size of the gains to a strategy to exploit a statistically significant result may be absolutely small or simply not great enough to outweigh transactions costs.

### LOS 11.f

The *p*-value for a hypothesis test is the smallest significance level for which the hypothesis would be rejected. For example, a *p*-value of 7% means the hypothesis can be rejected at the 10% significance level but cannot be rejected at the 5% significance level.

### LOS 11.g

With unknown population variance, the *t*-statistic is used for tests about the mean of a normally distributed population: $t_{n-1} = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$ . If the population variance is known, the appropriate test statistic is $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ for tests about the mean of a population.

## LOS 11.h

For two independent samples from two normally distributed populations, the difference in means can be tested with a *t*-statistic. When the two population variances are assumed to be equal, the denominator is based on the variance of the pooled samples, but when sample variances are assumed to be unequal, the denominator is based on a combination of the two samples' variances.

## LOS 11.i

A paired comparisons test is concerned with the mean of the differences between the paired observations of two dependent, normally distributed samples. A *t*-statistic, $t = \frac{\bar{d}}{s_{\bar{d}}}$, where $s_{\bar{d}} = \frac{s_d}{\sqrt{n}}$, and $\bar{d}$ is the average difference of the *n* paired observations, is used to test whether the means of two dependent normal variables are equal. Values outside the critical *t*-values lead us to reject equality.

## LOS 11.j

The test of a hypothesis about the population variance for a normally distributed population uses a chi-square test statistic: $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$ , where *n* is the sample size, $s^2$ is the sample variance, and $\sigma_0^2$ is the hypothesized value for the population variance. Degrees of freedom are n − 1.

The test comparing two variances based on independent samples from two normally distributed populations uses an F-distributed test statistic: $F = \frac{s_1^2}{s_2^2}$ , where $s_1^2$ is the variance of the first sample and $s_2^2$ is the (smaller) variance of the second sample.

## LOS 11.k

To test a hypothesis that a population correlation coefficient equals zero, the appropriate test statistic is a *t*-statistic with n – 2 degrees of freedom, calculated as

$\frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$ where *r* is the sample correlation coefficient.

## LOS 11.l

Parametric tests, like the *t*-test, *F*-test, and chi-square tests, make assumptions regarding the distribution of the population from which samples are drawn. Nonparametric tests either do not consider a particular population parameter or have few assumptions about the sampled population. Nonparametric tests are used when the assumptions of parametric tests can't be supported or when the data are not suitable for parametric tests.

# ANSWER KEY FOR MODULE QUIZZES

## Module Quiz 11.1

1. **C** To test whether the population mean is greater than 20, the test would attempt to reject the null hypothesis that the mean is less than or equal to 20. The null hypothesis must always include the "equal to" condition. (LOS 11.a)

2. **C** Rejecting the null when it is actually true is a Type I error. A Type II error is failing to reject the null hypothesis when it is false. The significance level equals the probability of a Type I error. (LOS 11.b, LOS 11.c)

3. **C** A Type I error is rejecting the null hypothesis when it's true. The probability of rejecting a false null is [1 − Prob Type II] = [1 − 0.60] = 40%, which is called the power of the test. A and B are not necessarily true, since the null may be false and the probability of rejection unknown. (LOS 11.c)

4. **A** The power of a test is 1 − P(Type II error) = 1 − 0.15 = 0.85. (LOS 11.d)

## Module Quiz 11.2

1. **B** With a *p*-value of 3%, the manager can reject the null hypothesis (that abnormal returns are less than or equal to zero) using a significance level of 3% or higher. Although the test result is statistically significant at significance levels as small as 3%, this does not necessarily imply that the result is economically meaningful. (LOS 11.e, LOS 11.f)

2. **A** With a small sample size, a *t*-test may be used if the population is approximately normally distributed. If the population has a nonnormal distribution, no test statistic is available unless the sample size is large. (LOS 11.g)

3. **B** This is a two-tailed test with 14 – 1 = 13 degrees of freedom. From the *t*-table, 2.160 is the critical value to which the analyst should compare the calculated *t*-statistic. (LOS 11.g)

## Module Quiz 11.3

1. **C** When the variances are assumed to be unequal, we just calculate the denominator (standard error) differently and use both sample variances to calculate the *t*-statistic. (LOS 11.h)

2. **A** Since the observations are likely dependent (both related to market returns), a paired comparisons (mean differences) test is appropriate and is based on a *t*-statistic. (LOS 11.h, LOS 11.i)

3. **C** There is no consistent relationship between the mean and standard deviation of the chi-square distribution or *F*-distribution. (LOS 11.j)

4. **B** The *F*-test is the appropriate test. (LOS 11.j)

5. **C** A test of the population variance is a chi-square test. (LOS 11.j)

# TOPIC ASSESSMENT: QUANTITATIVE METHODS

*You have now finished the Quantitative Methods topic section. The following Topic Assessment provides immediate feedback on how effective your study has been for this material. The number of questions on this test is equal to the number of questions for the topic on one-half of the actual Level I CFA exam. Questions are more exam-like than typical Module Quiz or QBank questions; a score of less than 70% indicates that your study likely needs improvement. These tests are best taken timed; allow 1.5 minutes per question.*

*After you've completed this Topic Assessment, you may additionally log in to your Schweser.com online account and enter your answers in the Topic Assessments product. Select "Performance Tracker" to view a breakdown of your score. Select "Compare with Others" to display how your score on the Topic Assessment compares to the scores of others who entered their answers.*

1. Allan Jabber invested $400 at the beginning of each of the last 12 months in the shares of a mutual fund that paid no dividends. Which method will he correctly choose to calculate his average price per share from the monthly share prices?
   A. Arithmetic mean.
   B. Harmonic mean.
   C. Geometric mean.

2. Colonia has only two political parties, the Wigs and the Wags. If the Wags are elected, there is a 32% probability of a tax increase over the next four years. If the Wigs are elected, there is a 60% probability of a tax increase. Based on the current polls, there is a 20% probability that the Wags will be elected. The sum of the (unconditional) probability of a tax increase and the joint probability that the Wigs will be elected and there will be no tax increase are *closest* to:
   A. 55%.
   B. 70%.
   C. 85%.

3. An analyst who wants to display the relationship between two variables graphically is *most likely* to use:
   A. a histogram.
   B. a scatterplot.
   C. a frequency polygon.

4. Ralph will retire 15 years from today and has saved $121,000 in his investment account for retirement. He believes he will need $37,000 at the beginning of each year for 25 years of retirement, with the first withdrawal on the day he retires. Ralph assumes that his investment account will return 8%. The amount he needs to deposit at the beginning of this year and each of the following 14 years (15 deposits in all) is *closest* to:
   A. $1,350.
   B. $1,450.
   C. $1,550.

5. The current price of Bosto shares is 50. Over the coming year, there is a 40% probability that share returns will be 10%, a 40% probability that share returns will be 12.5%, and a 20% probability that share returns will be 30%. Bosto's expected return

and standard deviation of returns for the coming year are *closest* to:

| | Expected return | Standard deviation |
|---|---|---|
| A. | 15.0% | 7.58% |
| B. | 17.5% | 5.75% |
| C. | 17.5% | 7.58% |

6. Nikki Ali and Donald Ankard borrowed $15,000 to finance their wedding and reception. The fully amortizing loan at 11% requires equal payments at the end of each of the next seven years. The principal portion of the first payment is *closest* to:
   A. $1,500.
   B. $1,530.
   C. $1,560.

7. Which of the following statements about probability distributions is *least accurate*?
   A. Continuous uniform distributions have cumulative distribution functions that are straight lines from zero to one.
   B. The probability that a continuously distributed random variable will take on a specific value is always zero.
   C. A normally distributed random variable divided by its standard deviation will follow a standard normal probability distribution.

8. An analyst wants to construct a hypothesis test to determine whether the mean weekly return on a stock is positive. The null hypothesis for this test should be that the mean return is:
   A. greater than zero.
   B. less than or equal to zero.
   C. greater than or equal to zero.

9. X, Y, and Z are independently distributed random variables. The probability of X is 30%, the probability of Y is 40%, and the probability of Z is 20%. Which of the following is *closest* to the probability that either X or Y will occur?
   A. 70%.
   B. 58%.
   C. 12%.

10. An analyst should use a *t*-test with n – 1 degrees of freedom to test a null hypothesis that two variables have:
    A. equal means.
    B. equal variances.
    C. no linear relationship.

11. The percentage changes in annual earnings for a company are approximately normally distributed with a mean of 5% and a standard deviation of 12%. The probability that the average change in earnings over the next five years will be greater than 15.5% is *closest* to:
    A. 2.5%.
    B. 5.0%.
    C. 10.0%.

12. Which of the following is *least likely* correct concerning a random variable that is lognormally distributed?
    A. It has a symmetric distribution.

B. The natural logarithms of the random variable are normally distributed.
C. It is a univariate distribution.

# TOPIC ASSESSMENT ANSWERS: QUANTITATIVE METHODS

1. **B** The harmonic mean of the 12 purchase prices will be his average price paid per share. (Study Session 2, Module 7.1, LOS 7.e)

2. **C** The unconditional probability of a tax increase is: 0.2(0.32) + 0.8(0.6) = 54.4%.The joint probability that the Wigs will be elected and there will be no tax increase is: 0.8(0.4) = 32%. The sum is: 54.4 + 32 = 86.4%. (Study Session 2, Module 8.1, LOS 8.f)

3. **B** Scatterplots illustrate the relationship between two variables. Histograms and frequency polygons show the distribution of observations for a single variable. (Study Session 2, Modules 7.1, 8.2, LOS 7.d, LOS 8.k)

4. **B** Step 1: Calculate the amount needed at retirement at t = 15, with your calculator in BGN mode.

   N = 25, FV = 0, I/Y = 8, PMT = 37,000, CPT PV = –426,564

   Step 2: Calculate the required deposits at t = 0,1,….,14 to result in a time 15 value of 426,564, with your calculator still in BGN mode.

   PV = –121,000, N = 15, I/Y = 8, FV = 426,564, CPT PMT = –$1,457.21 (Study Session 2, Module 6.2, LOS 6.e)

5. **A** E[R] = (0.4)(10) + (0.4)(12.5) + (0.2)(30) = 15%

   Variance = $(0.4)(10 - 15)^2 + (0.4)(12.5 - 15)^2 + (0.2)(30 - 15)^2 = 57.5$

   Standard deviation = $\sqrt{57.5}$ = 7.58%

   (Study Session 2, Module 7.2, LOS 7.g)

6. **B** The interest portion of the first payment is simply principal × interest rate = (15,000 × 0.11) = 1,650.

   Using a financial calculator: PV = 15,000, FV = 0, I/Y = 11, N = 7, CPT PMT= $3,183

   Principal = payment − interest = 3,183 − 1,650 = 1,533

   (Study Session 2, Module 6.2, LOS 6.f)

7. **C** A standard normal probability distribution has a mean of zero, so subtracting the mean from a normal random variable before dividing by its standard deviation is necessary to produce a standard normal probability distribution. (Study Session 3, Module 9.2, LOS 9.l)

8. **B** The null hypothesis should state the condition which, if rejected, would lend statistical support to the alternative hypothesis. Here, the null hypothesis should be that the mean return is less than or equal to zero, and the alternative hypothesis should be that the mean return is greater than zero. (Study Session 3, Module 11.1, LOS 11.a)

9. **B** The probability of X or Y is P(X) + P(Y) − P(XY).

   0.3 + 0.4 − (0.3)(0.4) = 58%

   (Study Session 3, Module 9.1, LOS 9.f)

10. **A** Difference-in-means hypothesis tests are *t*-tests with n – 1 degrees of freedom. Tests of correlation are *t*-tests with n – 2 degrees of freedom. Tests for equality of variances are *F*-tests. (Study Session 3, Module 11.3, LOS 11.h, LOS 11.j, LOS 11.k)

11. **A** The standard error of a 5-year average of earnings changes is $\frac{12\%}{\sqrt{5}}$ = 5.366%. 15.5% is $\frac{15.5-5}{5.366}$ = 1.96 standard errors above the mean, and the probability of a 5-year average more than 1.96 standard errors above the mean is 2.5% for a normal distribution. (Study Session 3, Module 10.1, LOS 10.f)

12. **A** A lognormal distribution is skewed to the right (positively skewed). (Study Session 3, Module 9.3, LOS 9.n)

# FORMULAS

nominal risk-free rate = real risk-free rate + expected inflation rate

required interest rate on a security = nominal risk-free rate
+ default risk premium
+ liquidity premium
+ maturity risk premium

effective annual rate = $(1 + \text{periodic rate})^m - 1$

continuous compounding: $e^r - 1 = \text{EAR}$

$$PV_{perpetuity} = \frac{PMT}{I/Y}$$

$FV = PV(1 + I/Y)^N$

population mean: $\mu = \frac{\sum_{i=1}^{N} X_i}{N}$

sample mean: $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$

geometric mean return ($R_G$): $1 + R_G = \sqrt[n]{(1+R_1) \times (1+R_2) \times ... \times (1+R_n)}$

harmonic mean: $\overline{X}_H = \frac{N}{\sum_{i=1}^{N} \frac{1}{x_i}}$

weighted mean: $\overline{X}_W = \sum_{i=1}^{n} w_i X_i$

position of the observation at a given percentile, *y*: $L_y = (n+1)\frac{y}{100}$

range = maximum value − minimum value

excess kurtosis = sample kurtosis − 3

$$MAD = \frac{\sum_{i=1}^{n} |X_i - \overline{X}|}{n}$$

population variance = $\sigma^2 = \frac{\sum_{i=1}^{N} \left(X_i - \mu\right)^2}{N}$,

where μ = population mean and N = number of possible outcomes

$$\text{sample variance} = s^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1},$$

where $\overline{X}$ = sample mean and n = sample size

coefficient of variation: $CV = \frac{s_x}{\overline{X}} = \frac{\text{standard deviation of x}}{\text{average value of x}}$

joint probability: $P(AB) = P(A \mid B) \times P(B)$

addition rule: $P(A \text{ or } B) = P(A) + P(B) - P(AB)$

multiplication rule: $P(A \text{ and } B) = P(A) \times P(B)$

total probability rule: $P(R) = P(R \mid S_1) \times P(S_1) + P(R \mid S_2) \times P(S_2) + \ldots + P(R \mid S_N) \times P(S_N)$

expected value: $E(X) = \Sigma P(x_i)x_i = P(x_1)x_1 + P(x_2)x_2 + \ldots + P(x_n)x_n$

$Cov(R_i,R_j) = E\{[R_i - E(R_i)][R_j - E(R_j)]\}$

$\text{Corr}\left(R_i, R_j\right) = \frac{Cov(R_i,R_j)}{\sigma(R_i)\sigma(R_j)}$

portfolio expected return: $E\left(R_p\right) = \sum_{i=1}^{N} w_i E\left(R_i\right) = w_1E(R_1) + w_2E(R_2) + \ldots + w_nE(R_n)$

portfolio variance: $\text{Var}\left(R_p\right) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \text{Cov}\left(R_i, R_j\right)$

where $w_i = \frac{\text{market value of investment in asset i}}{\text{market value of the portfolio}}$

Bayes' formula:

$\text{updated probability} = \frac{\text{probability of new information for a given event}}{\text{unconditional probability of new information}} \times \text{prior probability of event}$

combination (binomial) formula: ${}_nC_r = \frac{n!}{(n-r)!r!}$

permutation formula: ${}_nP_r = \frac{n!}{(n-r)!}$

binomial probability: $p(x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$

for a binomial random variable: $E(X) = np$; variance $= np(1 - p)$

for a normal variable:

90% confidence interval for X is $\overline{X} - 1.65s$ to $\overline{X} + 1.65s$

95% confidence interval for X is $\overline{X} - 1.96s$ to $\overline{X} + 1.96s$

99% confidence interval for X is $\overline{X} - 2.58s$ to $\overline{X} + 2.58s$

$z = \frac{\text{observation} - \text{population mean}}{\text{standard deviation}} = \frac{x-\mu}{\sigma}$

$\text{SFRatio} = \frac{[E(R_p) - R_L]}{\sigma_p}$

continuously compounded rate of return: $rcc = \ln\left(\frac{S_1}{S_0}\right) = \ln\left(1 + HPR\right)$

for a uniform distribution: $P\left(x_1 \leq X \leq x_2\right) = \frac{(x_2 - x_1)}{(b-a)}$

sampling error of the mean = sample mean – population mean = $\bar{x} - \mu$

standard error of the sample mean, known population variance: $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$

standard error of the sample mean, unknown population variance: $s_{\bar{x}} = \frac{s}{\sqrt{n}}$

confidence interval: point estimate ± (reliability factor × standard error)

confidence interval for the population mean: $\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

tests for population mean = $\mu_0$: z-statistic $= \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ , *t*-statistic $= \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$

test for equality of variances: $F = \frac{s_1^2}{s_2^2}$ , where $s_1^2 > s_2^2$

paired comparisons test: *t*-statistic = $\frac{\bar{d} - \mu_{dz}}{s_{\bar{d}}}$

test for differences in means:

*t*-statistic = $\frac{(x_1 - x_2)}{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^{1/2}}$ (sample variances assumed unequal)

*t*-statistic = $\frac{(x_1 - x_2)}{\left(\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}\right)^{1/2}}$ (sample variances assumed equal)

test for correlation: t = $\frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$

# APPENDICES

## APPENDIX A: AREAS UNDER THE NORMAL CURVE

Most of the examples in this book have used one version of the *z*-table to find the area under the normal curve. This table provides the cumulative probabilities (or the area under the entire curve to left of the z-value).

### Probability Example

Assume that the annual earnings per share (EPS) for a large sample of firms is normally distributed with a mean of \$5.00 and a standard deviation of \$1.50. What is the approximate probability of an observed EPS value falling between \$3.00 and \$7.25?

If $EPS = x = \$7.25$, then $z = (x - \mu)/\sigma = (\$7.25 - \$5.00)/\$1.50 = +1.50$

If $EPS = x = \$3.00$, then $z = (x - \mu)/\sigma = (\$3.00 - \$5.00)/\$1.50 = -1.33$

### Solving Using The Cumulative *Z*-Table

*For z-value of 1.50:* Use the row headed 1.5 and the column headed 0 to find the value 0.9332. This represents the area under the curve to the left of the critical value 1.50.

*For z-value of –1.33:* Use the row headed 1.3 and the column headed 3 to find the value 0.9082. This represents the area under the curve to the left of the critical value +1.33. The area to the left of –1.33 is 1 − 0.9082 = 0.0918.

The area between these critical values is 0.9332 − 0.0918 = 0.8414, or 84.14%.

### Hypothesis Testing—One-Tailed Test Example

A sample of a stock's returns on 36 nonconsecutive days results in a mean return of 2.0%. Assume the population standard deviation is 20.0%. Can we say with 95% confidence that the mean return is greater than 0%?

$H_0$: $\mu \leq 0.0\%$, $H_a$: $\mu > 0.0\%$. The test statistic = *z*-statistic

$= \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = (2.0 - 0.0) / (20.0 / 6) = 0.60$

The significance level = 1.0 − 0.95 = 0.05, or 5%. Because we are interested in a return greater than 0.0%, this is a one-tailed test.

### Using the Cumulative *Z*-Table

Because this is a one-tailed test with an alpha of 0.05, we need to find the value 0.95 in the cumulative *z*-table. The closest value is 0.9505, with a corresponding critical *z*-value of 1.65. Because the test statistic is less than the critical value, we fail to reject $H_0$.

### Hypothesis Testing—Two-Tailed Test Example

Using the same assumptions as before, suppose that the analyst now wants to determine if he can say with 99% confidence that the stock's return is not equal to 0.0%.

$H_0$: μ = 0.0%, $H_a$: μ ≠ 0.0%. The test statistic (*z*-value) = (2.0 − 0.0) / (20.0 / 6) = 0.60. The significance level = 1.0 − 0.99 = 0.01, or 1%. Because we are interested in whether or not the stock return is nonzero, this is a two-tailed test.

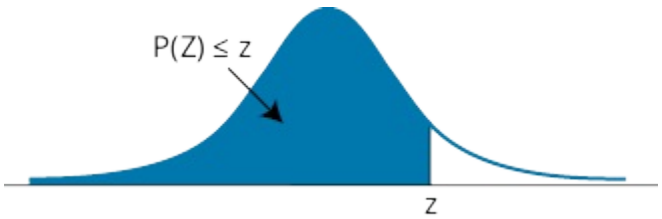
## Using the Cumulative *Z*-Table

Because this is a two-tailed test with an alpha of 0.01, there is a 0.005 rejection region in both tails. Thus, we need to find the value 0.995 (1.0 − 0.005) in the table. The closest value is 0.9951, which corresponds to a critical *z*-value of 2.58. Because the test statistic is less than the critical value, we fail to reject $H_0$ and conclude that the stock's return equals 0.0%.

## CUMULATIVE *Z*-TABLE

**Standard Normal Distribution**

P(Z ≤ z) = N(z) for z ≥ 0



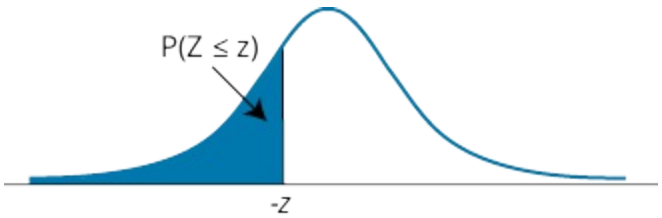

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| **0.1** | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| **0.2** | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| **0.3** | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| **0.4** | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| | | | | | | | | | | |
| **0.5** | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| **0.6** | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| **0.7** | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| **0.8** | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| **0.9** | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| | | | | | | | | | | |
| **1.0** | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| **1.1** | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| **1.2** | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| **1.3** | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| **1.4** | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| | | | | | | | | | | |
| **1.5** | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| **1.6** | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| **1.7** | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| **1.8** | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **1.9** | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| | | | | | | | | | | |
| **2.0** | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| **2.1** | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| **2.2** | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| **2.3** | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| **2.4** | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| | | | | | | | | | | |
| **2.5** | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| **2.6** | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| **2.7** | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| **2.8** | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| **2.9** | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| **3.0** | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |

# CUMULATIVE *Z*-TABLE (CONT.)

**Standard Normal Distribution**

$P(Z \le z) = N(z)$ for $z \ge 0$




| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |
| **–0.1** | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| **–0.2** | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| **–0.3** | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| **–0.4** | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| | | | | | | | | | | |
| **–0.5** | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| **–0.6** | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| **–0.7** | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2297 | 0.2266 | 0.2236 | 0.2207 | 0.2177 | 0.2148 |
| **–0.8** | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| **–0.9** | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| | | | | | | | | | | |
| | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **–1.0** | | | | | | | | | | |
| **–1.1** | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| **–1.2** | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1057 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| **–1.3** | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| **–1.4** | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| | | | | | | | | | | |
| **–1.5** | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| **–1.6** | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| **–1.7** | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| **–1.8** | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| **–1.9** | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| | | | | | | | | | | |
| **–2.0** | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| **–2.1** | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| **–2.2** | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| **–2.3** | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| **–2.4** | 0.0082 | 0.0080 | 0.0078 | 0.0076 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| | | | | | | | | | | |
| **–2.5** | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| **–2.6** | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| **–2.7** | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| **–2.8** | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| **–2.9** | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| **–3.0** | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |

# APPENDIX B: STUDENT'S *t*-DISTRIBUTION

| **Level of Significance for One-Tailed Test** | | | | | | |
|---|---|---|---|---|---|---|
| **df** | **0.100** | **0.050** | **0.025** | **0.01** | **0.005** | **0.0005** |

| | Level of Significance for Two-Tailed Test | | | | | |
|---|---|---|---|---|---|---|
| **df** | **0.20** | **0.10** | **0.05** | **0.02** | **0.01** | **0.001** |
| **1** | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 | 636.619 |
| **2** | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 | 31.599 |
| **3** | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 | 12.294 |
| **4** | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 | 8.610 |
| **5** | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 | 6.869 |
| | | | | | | |
| **6** | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 5.959 |
| **7** | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | 5.408 |
| **8** | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | 5.041 |
| **9** | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 4.781 |
| **10** | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | 4.587 |
| | | | | | | |
| **11** | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 | 4.437 |
| **12** | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 | 4.318 |
| **13** | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 | 4.221 |
| **14** | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 | 4.140 |
| **15** | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 | 4.073 |
| | | | | | | |
| **16** | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 | 4.015 |
| **17** | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 | 3.965 |
| **18** | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 | 3.922 |
| **19** | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 | 3.883 |
| **20** | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 | 3.850 |
| | | | | | | |
| **21** | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 | 3.819 |
| **22** | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 | 3.792 |
| **23** | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 | 3.768 |
| **24** | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 | 3.745 |
| **25** | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 | 3.725 |
| | | | | | | |
| **26** | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 | 3.707 |
| **27** | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 | 3.690 |
| **28** | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 | 3.674 |
| **29** | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 | 3.659 |
| **30** | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 | 3.646 |
| | | | | | | |
| **40** | 1.303 | 1.684 | 2.021 | 2.423 | 2.704 | 3.551 |
| **60** | 1.296 | 1.671 | 2.000 | 2.390 | 2.660 | 3.460 |
| **120** | 1.289 | 1.658 | 1.980 | 2.358 | 2.617 | 3.373 |
| ∞ | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 | 3.291 |

# APPENDIX C: *F*-TABLE AT 5% (UPPER TAIL)

***F*-Table, Critical Values, 5% in Upper Tail**

Degrees of freedom for the numerator along top row

Degrees of freedom for the denominator along side row

| | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **12** | **15** | **20** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 161 | 200 | 216 | 225 | 230 | 234 | 237 | 239 | 241 | 242 | 244 | 246 | 248 |
| **2** | 18.5 | 19.0 | 19.2 | 19.2 | 19.3 | 19.3 | 19.4 | 19.4 | 19.4 | 19.4 | 19.4 | 19.4 | 19.4 |
| **3** | 10.1 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 | 8.79 | 8.74 | 8.70 | 8.66 |
| **4** | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 | 5.96 | 5.91 | 5.86 | 5.80 |
| **5** | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 | 4.74 | 4.68 | 4.62 | 4.56 |
| | | | | | | | | | | | | | |
| **6** | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 | 4.06 | 4.00 | 3.94 | 3.87 |
| **7** | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 | 3.64 | 3.57 | 3.51 | 3.44 |
| **8** | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 | 3.35 | 3.28 | 3.22 | 3.15 |
| **9** | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 | 3.14 | 3.07 | 6.01 | 2.94 |
| **10** | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 | 2.98 | 2.91 | 2.85 | 2.77 |
| | | | | | | | | | | | | | |
| **11** | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 | 2.95 | 2.90 | 2.85 | 2.79 | 2.72 | 2.65 |
| **12** | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 | 2.75 | 2.69 | 2.62 | 2.54 |
| **13** | 4.67 | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.83 | 2.77 | 2.71 | 2.67 | 2.60 | 2.53 | 2.46 |
| **14** | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.76 | 2.70 | 2.65 | 2.60 | 2.53 | 2.46 | 2.39 |
| **15** | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 | 2.54 | 2.48 | 2.40 | 2.33 |
| | | | | | | | | | | | | | |
| **16** | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 | 2.59 | 2.54 | 2.49 | 2.42 | 2.35 | 2.28 |
| **17** | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.61 | 2.55 | 2.49 | 2.45 | 2.38 | 2.31 | 2.23 |
| **18** | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 | 2.51 | 2.46 | 2.41 | 2.34 | 2.27 | 2.19 |
| **19** | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.54 | 2.48 | 2.42 | 2.38 | 2.31 | 2.23 | 2.16 |
| **20** | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 | 2.35 | 2.28 | 2.20 | 2.12 |
| | | | | | | | | | | | | | |
| **21** | 4.32 | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.49 | 2.42 | 2.37 | 2.32 | 2.25 | 2.18 | 2.10 |
| **22** | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.46 | 2.40 | 2.34 | 2.30 | 2.23 | 2.15 | 2.07 |
| **23** | 4.28 | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.44 | 2.37 | 2.32 | 2.27 | 2.20 | 2.13 | 2.05 |
| **24** | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.42 | 2.36 | 2.30 | 2.25 | 2.18 | 2.11 | 2.03 |
| **25** | 4.24 | 3.39 | 2.99 | 2.76 | 2.60 | 2.49 | 2.40 | 2.34 | 2.28 | 2.24 | 2.16 | 2.09 | 2.01 |
| | | | | | | | | | | | | | |
| **30** | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.33 | 2.27 | 2.21 | 2.16 | 2.09 | 2.01 | 1.93 |
| **40** | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.25 | 2.18 | 2.12 | 2.08 | 2.00 | 1.92 | 1.84 |
| **60** | 4.00 | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.17 | 2.10 | 2.04 | 1.99 | 1.92 | 1.84 | 1.75 |
| **120** | 3.92 | 3.07 | 2.68 | 2.45 | 2.29 | 2.18 | 2.09 | 2.02 | 1.96 | 1.91 | 1.83 | 1.75 | 1.66 |
| ∞ | 3.84 | 3.00 | 2.60 | 2.37 | 2.21 | 2.10 | 2.01 | 1.94 | 1.88 | 1.83 | 1.75 | 1.67 | 1.57 |

# APPENDIX D: *F*-TABLE AT 2.5% (UPPER TAIL)

### *F*-Table, Critical Values, 2.5% in Upper Tails

Degrees of freedom for the numerator along top row

Degrees of freedom for the denominator along side row

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 648 | 799 | 864 | 900 | 922 | 937 | 948 | 957 | 963 | 969 | 977 |
| **2** | 38.51 | 39.00 | 39.17 | 39.25 | 39.30 | 39.33 | 39.36 | 39.37 | 39.39 | 39.40 | 39.41 |
| **3** | 17.44 | 16.04 | 15.44 | 15.10 | 14.88 | 14.73 | 14.62 | 14.54 | 14.47 | 14.42 | 14.34 |
| **4** | 12.22 | 10.65 | 9.98 | 9.60 | 9.36 | 9.20 | 9.07 | 8.98 | 8.90 | 8.84 | 8.75 |
| **5** | 10.01 | 8.43 | 7.76 | 7.39 | 7.15 | 6.98 | 6.85 | 6.76 | 6.68 | 6.62 | 6.52 |
| | | | | | | | | | | | |
| **6** | 8.81 | 7.26 | 6.60 | 6.23 | 5.99 | 5.82 | 5.70 | 5.60 | 5.52 | 5.46 | 5.37 |
| **7** | 8.07 | 6.54 | 5.89 | 5.52 | 5.29 | 5.12 | 4.99 | 4.90 | 4.82 | 4.76 | 4.67 |
| **8** | 7.57 | 6.06 | 5.42 | 5.05 | 4.82 | 4.65 | 4.53 | 4.43 | 4.36 | 4.30 | 4.20 |
| **9** | 7.21 | 5.71 | 5.08 | 4.72 | 4.48 | 4.32 | 4.20 | 4.10 | 4.03 | 3.96 | 3.87 |
| **10** | 6.94 | 5.46 | 4.83 | 4.47 | 4.24 | 4.07 | 3.95 | 3.85 | 3.78 | 3.72 | 3.62 |
| | | | | | | | | | | | |
| **11** | 6.72 | 5.26 | 4.63 | 4.28 | 4.04 | 3.88 | 3.76 | 3.66 | 3.59 | 3.53 | 3.43 |
| **12** | 6.55 | 5.10 | 4.47 | 4.12 | 3.89 | 3.73 | 3.61 | 3.51 | 3.44 | 3.37 | 3.28 |
| **13** | 6.41 | 4.97 | 4.35 | 4.00 | 3.77 | 3.60 | 3.48 | 3.39 | 3.31 | 3.25 | 3.15 |
| **14** | 6.30 | 4.86 | 4.24 | 3.89 | 3.66 | 3.50 | 3.38 | 3.29 | 3.21 | 3.15 | 3.05 |
| **15** | 6.20 | 4.77 | 4.15 | 3.80 | 3.58 | 3.41 | 3.29 | 3.20 | 3.12 | 3.06 | 2.96 |
| | | | | | | | | | | | |
| **16** | 6.12 | 4.69 | 4.08 | 3.73 | 3.50 | 3.34 | 3.22 | 3.12 | 3.05 | 2.99 | 2.89 |
| **17** | 6.04 | 4.62 | 4.01 | 3.66 | 3.44 | 3.28 | 3.16 | 3.06 | 2.98 | 2.92 | 2.82 |
| **18** | 5.98 | 4.56 | 3.95 | 3.61 | 3.38 | 3.22 | 3.10 | 3.01 | 2.93 | 2.87 | 2.77 |
| **19** | 5.92 | 4.51 | 3.90 | 3.56 | 3.33 | 3.17 | 3.05 | 2.96 | 2.88 | 2.82 | 2.72 |
| **20** | 5.87 | 4.46 | 3.86 | 3.51 | 3.29 | 3.13 | 3.01 | 2.91 | 2.84 | 2.77 | 2.68 |
| | | | | | | | | | | | |
| **21** | 5.83 | 4.42 | 3.82 | 3.48 | 3.25 | 3.09 | 2.97 | 2.87 | 2.80 | 2.73 | 2.64 |
| **22** | 5.79 | 4.38 | 3.78 | 3.44 | 3.22 | 3.05 | 2.93 | 2.84 | 2.76 | 2.70 | 2.60 |
| **23** | 5.75 | 4.35 | 3.75 | 3.41 | 3.18 | 3.02 | 2.90 | 2.81 | 2.73 | 2.67 | 2.57 |
| **24** | 5.72 | 4.32 | 3.72 | 3.38 | 3.15 | 2.99 | 2.87 | 2.78 | 2.70 | 2.64 | 2.54 |
| **25** | 5.69 | 4.29 | 3.69 | 3.35 | 3.13 | 2.97 | 2.85 | 2.75 | 2.68 | 2.61 | 2.51 |
| | | | | | | | | | | | |
| **30** | 5.57 | 4.18 | 3.59 | 3.25 | 3.03 | 2.87 | 2.75 | 2.65 | 2.57 | 2.51 | 2.41 |
| **40** | 5.42 | 4.05 | 3.46 | 3.13 | 2.90 | 2.74 | 2.62 | 2.53 | 2.45 | 2.39 | 2.29 |
| **60** | 5.29 | 3.93 | 3.34 | 3.01 | 2.79 | 2.63 | 2.51 | 2.41 | 2.33 | 2.27 | 2.17 |
| **120** | 5.15 | 3.80 | 3.23 | 2.89 | 2.67 | 2.52 | 2.39 | 2.30 | 2.22 | 2.16 | 2.05 |
| ∞ | 5.02 | 3.69 | 3.12 | 2.79 | 2.57 | 2.41 | 2.29 | 2.19 | 2.11 | 2.05 | 1.94 |

# APPENDIX E: CHI-SQUARED TABLE

### Values of χ2 (Degrees of Freedom, Level of Significance)

### Probability in Right Tail

| Degrees of Freedom | 0.99 | 0.975 | 0.95 | 0.9 | 0.1 | 0.05 | 0.025 | 0.01 |
|---|---|---|---|---|---|---|---|---|
| **1** | 0.000157 | 0.000982 | 0.003932 | 0.0158 | 2.706 | 3.841 | 5.024 | 6.635 |
| **2** | 0.020100 | 0.050636 | 0.102586 | 0.2107 | 4.605 | 5.991 | 7.378 | 9.210 |
| | | | | | | | | |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **3** | 0.1148 | 0.2158 | 0.3518 | 0.5844 | 6.251 | 7.815 | 9.348 | 11.345 |
| **4** | 0.297 | 0.484 | 0.711 | 1.064 | 7.779 | 9.488 | 11.143 | 13.277 |
| **5** | 0.554 | 0.831 | 1.145 | 1.610 | 9.236 | 11.070 | 12.832 | 15.086 |
| | | | | | | | | |
| **6** | 0.872 | 1.237 | 1.635 | 2.204 | 10.645 | 12.592 | 14.449 | 16.812 |
| **7** | 1.239 | 1.690 | 2.167 | 2.833 | 12.017 | 14.067 | 16.013 | 18.475 |
| **8** | 1.647 | 2.180 | 2.733 | 3.490 | 13.362 | 15.507 | 17.535 | 20.090 |
| **9** | 2.088 | 2.700 | 3.325 | 4.168 | 14.684 | 16.919 | 19.023 | 21.666 |
| **10** | 2.558 | 3.247 | 3.940 | 4.865 | 15.987 | 18.307 | 20.483 | 23.209 |
| | | | | | | | | |
| **11** | 3.053 | 3.816 | 4.575 | 5.578 | 17.275 | 19.675 | 21.920 | 24.725 |
| **12** | 3.571 | 4.404 | 5.226 | 6.304 | 18.549 | 21.026 | 23.337 | 26.217 |
| **13** | 4.107 | 5.009 | 5.892 | 7.041 | 19.812 | 22.362 | 24.736 | 27.688 |
| **14** | 4.660 | 5.629 | 6.571 | 7.790 | 21.064 | 23.685 | 26.119 | 29.141 |
| **15** | 5.229 | 6.262 | 7.261 | 8.547 | 22.307 | 24.996 | 27.488 | 30.578 |
| | | | | | | | | |
| **16** | 5.812 | 6.908 | 7.962 | 9.312 | 23.542 | 26.296 | 28.845 | 32.000 |
| **17** | 6.408 | 7.564 | 8.672 | 10.085 | 24.769 | 27.587 | 30.191 | 33.409 |
| **18** | 7.015 | 8.231 | 9.390 | 10.865 | 25.989 | 28.869 | 31.526 | 34.805 |
| **19** | 7.633 | 8.907 | 10.117 | 11.651 | 27.204 | 30.144 | 32.852 | 36.191 |
| **20** | 8.260 | 9.591 | 10.851 | 12.443 | 28.412 | 31.410 | 34.170 | 37.566 |
| | | | | | | | | |
| **21** | 8.897 | 10.283 | 11.591 | 13.240 | 29.615 | 32.671 | 35.479 | 38.932 |
| **22** | 9.542 | 10.982 | 12.338 | 14.041 | 30.813 | 33.924 | 36.781 | 40.289 |
| **23** | 10.196 | 11.689 | 13.091 | 14.848 | 32.007 | 35.172 | 38.076 | 41.638 |
| **24** | 10.856 | 12.401 | 13.848 | 15.659 | 33.196 | 36.415 | 39.364 | 42.980 |
| **25** | 11.524 | 13.120 | 14.611 | 16.473 | 34.382 | 37.652 | 40.646 | 44.314 |
| | | | | | | | | |
| **26** | 12.198 | 13.844 | 15.379 | 17.292 | 35.563 | 38.885 | 41.923 | 45.642 |
| **27** | 12.878 | 14.573 | 16.151 | 18.114 | 36.741 | 40.113 | 43.195 | 46.963 |
| **28** | 13.565 | 15.308 | 16.928 | 18.939 | 37.916 | 41.337 | 44.461 | 48.278 |
| **29** | 14.256 | 16.047 | 17.708 | 19.768 | 39.087 | 42.557 | 45.722 | 49.588 |
| **30** | 14.953 | 16.791 | 18.493 | 20.599 | 40.256 | 43.773 | 46.979 | 50.892 |
| | | | | | | | | |
| **50** | 29.707 | 32.357 | 34.764 | 37.689 | 63.167 | 67.505 | 71.420 | 76.154 |
| **60** | 37.485 | 40.482 | 43.188 | 46.459 | 74.397 | 79.082 | 83.298 | 88.379 |
| **80** | 53.540 | 57.153 | 60.391 | 64.278 | 96.578 | 101.879 | 106.629 | 112.329 |
| **100** | 70.065 | 74.222 | 77.929 | 82.358 | 118.498 | 124.342 | 129.561 | 135.807 |

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